Pendulum (Simplified Problem).

$$\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

$$\left.\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dx} &= -\gamma y - \omega^2 \sin x \end{aligned}\right\} \text{ General system.}$$

Simplified system:- $\gamma = 1, \quad \omega = 1$

$$\Rightarrow \quad \frac{dx}{dy} = y \qquad = F(x,y)$$

$$\frac{dy}{dt} = -y - \sin x \qquad = G(x,y)$$

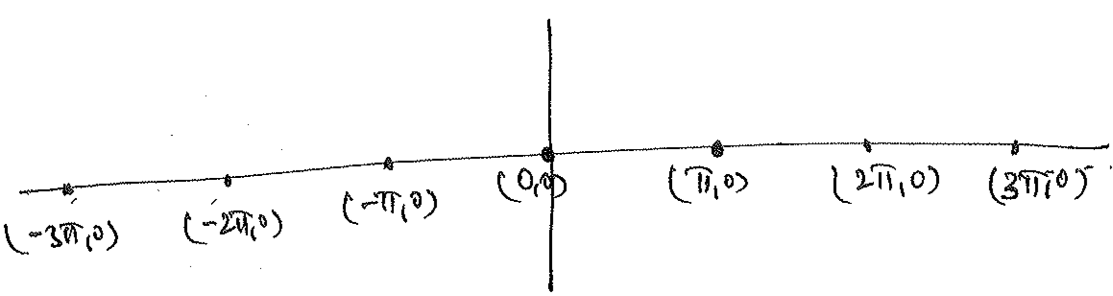
Critical Points: $F(x,y) = 0 \;\&\; G(x,y) = 0$

$\Rightarrow \quad y = 0 \quad \& \quad -y - \sin x = 0$

Using $y = 0$, we have $\sin x = 0$

$\Rightarrow x = 0, \pm\pi, \pm 2\pi, \dots$

$\Rightarrow$ System has infinite number of critical points.



Let us analyse the system at three critical points:
$(-\pi,0), \; (0,0), \; (\pi,0)$.

Since $F(x,y) = y$

$G(x,y) = -y - \sin x$

$F_x = 0 \qquad F_y = 1$

$G_x = -\cos x \qquad G_y = -1$

$$\Rightarrow J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos x & -1 \end{bmatrix}$$

Recall that near the critical point (x_0, y_0), the locally linear system is $\vec{u}' = J_{(x_0,y_0)}\vec{u}$ where $\vec{u} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$

Locally linear system near $(0,0)$:-

$$J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\Rightarrow \vec{u}' = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \vec{u}$$

Eigenvalues: $(-\lambda)(-1-\lambda) + 1 = 0 \Rightarrow \lambda^2 + \lambda + 1 = 0$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$

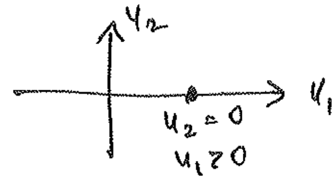
$$= \frac{-1 \pm i\sqrt{3}}{2}$$: Complex conjugates.

$\Rightarrow$ Spiral solutions.

~~Eigenvectors:~~

Direction of Spirals $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\Rightarrow \quad u_1' = u_2 \qquad \text{--- (i)}$$
$$u_2' = -u_1 - u_2 \qquad \text{--- (ii)}$$



Choose a point $u_2 = 0,\ u_1 > 0$

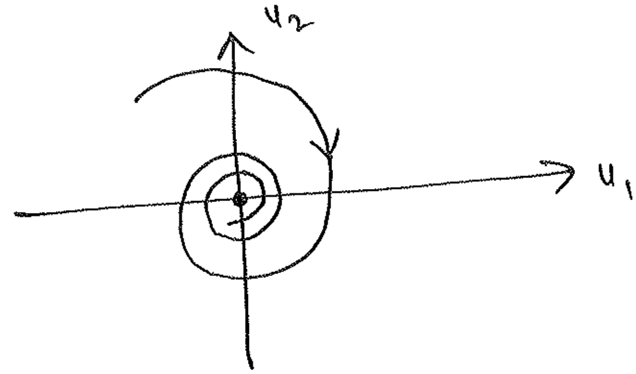
from (ii), $\quad u_2' = -u_1 < 0 \quad \Rightarrow \quad u_2' < 0$

$\Rightarrow$ u_2 decreasing here with time

$\Rightarrow$ Spiral is clockwise.

Also real part of $(\lambda) < 0$

$\Rightarrow$ Spiral is decaying.



Near $(\pi, 0)$:

$$J = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \quad \vec{u}' = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \vec{u} \end{bmatrix}$$

Eigenvalues: $\quad (-\lambda)(-1-\lambda) - 1 = 0 \quad \Rightarrow \quad \lambda^2 + \lambda - 1 = 0$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1+4}}{2}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$

$$\therefore \lambda_1 = \frac{-1+\sqrt{5}}{2} > 0, \quad \lambda_2 = \frac{-1-\sqrt{5}}{2} < 0$$

$\Rightarrow$ Saddle.

Eigenvectors: with $\lambda_1 = \frac{-1+\sqrt{5}}{2}$, $\cancel{\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}$

$\cancel{(\mathbf{J}-\lambda\mathbf{I})\vec{x}=0}$ $\quad (\mathbf{J}-\lambda_1 \mathbf{I})\vec{v}=0$

$$\Rightarrow \begin{bmatrix} -\lambda_1 & 1 \\ 1 & -1-\lambda_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0$$

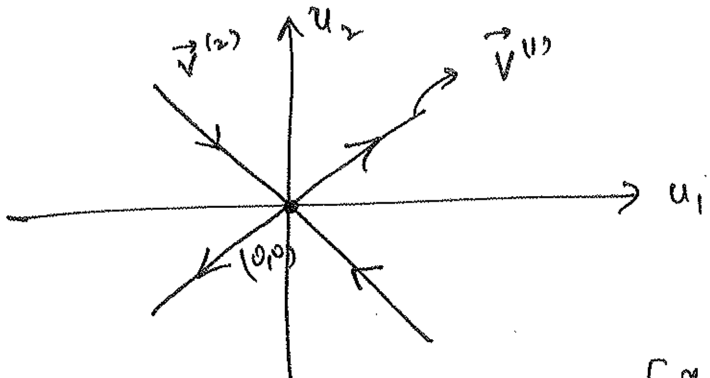
$$\Rightarrow \begin{bmatrix} -\lambda_1 v_1 + v_2 \\ v_1-(1+\lambda_1)v_2 \end{bmatrix}=0$$

$$-\lambda_1 v_1 + v_2 = 0 \quad \Rightarrow \quad v_2=\lambda_1 v_1$$

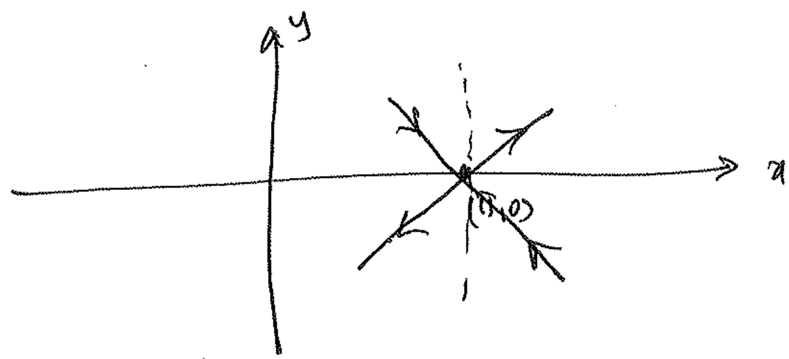
Choosing $v_1=1,\ v_2=\lambda_1 \quad \Rightarrow \vec{v}^{(1)}=\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$

with $\underline{\underline{\lambda=\lambda_2}}$! we get $\vec{v}^{(2)}=\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

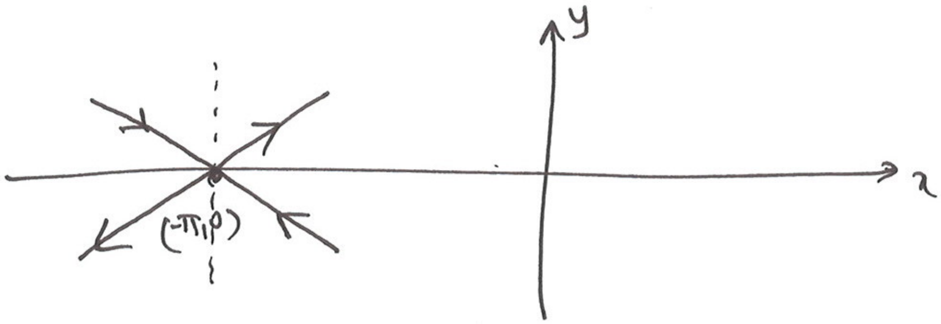
Recall $\lambda_1>0,\ \lambda_2<0$



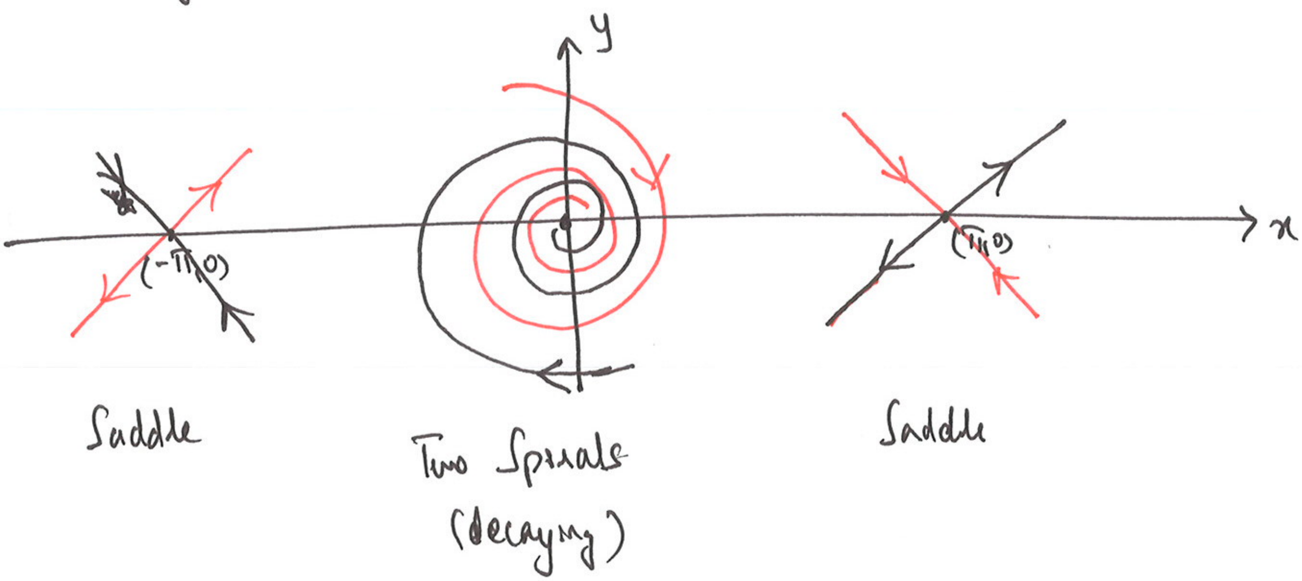
Since $\vec{u}=\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}=\begin{bmatrix} x-\pi \\ y-0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} u_1+\pi \\ u_2 \end{bmatrix}$



Similarly, near $(-\pi, 0)$, we get



Combining the three "local" phase portraits.



Global phase portrait :- Connect the curves in a sensible way.