Nonhomogeneous Linear Systems: Method of Undetermined Coefficients

$$\vec{X}' = A\vec{X} + \vec{g}$$: To find particular solution:

General tricks:

- If $\vec{g}$ involves exponential ~~terms~~ terms, try a particular solution of the form $\vec{X}_p = \vec{v}[\text{exponential term}]$.

- If the exponential term is part of the homogeneous solution, the try $\vec{X}_p = t\,\vec{v}\,[\text{exponential term}] + \vec{\eta}\,[\text{exponential term}]$

- If $\vec{g}$ has a cos or sine term, solve the more general problem with the nonhomogeneous term replaced with $e^{i\omega t}$. Recall that $\cos\omega t = \mathrm{Re}[e^{i\omega t}]$
$\sin\omega t = \mathrm{Im}[e^{i\omega t}]$.

- If $\vec{g}$ is constant, then let $\vec{X}_p = \vec{V}$ ↳ also a constant vector.

The goal in each case is to find $\vec{v}$.

Let us see a few examples to clarify the above points.

Ex! Constant $\vec{g}$ case!

$$\vec{X}' = \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix} \vec{X} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix}$$

find the particular solution. $\qquad \vec{g} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution: Since $\vec{g} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a constant vector, we try

$$\vec{X}_p = \vec{V} \longrightarrow \text{constant vector.}$$

$$\therefore \vec{X}_p' = 0$$

Substituting $\vec{X}_p$ in $\vec{X}' = A\vec{X} + \vec{g}$, we get

$$0 = A\vec{V} + \vec{g} \Rightarrow A\vec{V} = -\vec{g}$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \vec{V} \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{V} = -\begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{\begin{bmatrix} -2 & -2 \\ -1 & -2 \end{bmatrix}}{4-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} -2 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -4 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/2 \end{bmatrix}$$

$$\therefore \vec{X}_p = \begin{bmatrix} 2 \\ 3/2 \end{bmatrix}$$

Homogeneous Solution! - $\vec{X}' = \begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix} \vec{X}$

Eigenvalues! $|A - \lambda I| = 0 \Rightarrow (-2-\lambda)(-2-\lambda) - 2 = 0$

$\Rightarrow \quad r^2 + 4r + 2 = 0$

$\Rightarrow \quad r_1 = -2-\sqrt{2}$

$r_2 = \sqrt{2}-2$

Eigenvectors: with $r_1 = -2-\sqrt{2}$, $\vec{\xi}^{(1)} = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$

with $r_2 = \sqrt{2}-2$; $\vec{\xi} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$

$\therefore$ Homogeneous solution : $\vec{X}_H = c_1 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{(-2-\sqrt{2})t} + c_2 \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} e^{(\sqrt{2}-2)t}$

General Solution, $\vec{X} = \vec{X}_H + \vec{X}_p$

Ex: Constant $\vec{g}$ case : Second type of Solution :

$$\vec{x}' = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Check determinant first : Here $|A| = 2-2 = 0$

  $\therefore$ If $\vec{X}_p = \vec{v}$, then we get $A\vec{v} + \vec{g} = 0$

  $\Rightarrow \vec{v} = -A^{-1}\vec{g}$

  Since $|A| = 0$, we cannot calculate A^{-1}.

- Try $\vec{X}_p = \vec{v}t + \vec{\eta}$

  Then $\vec{X}_p' = \vec{v}$

  Substitute $\vec{X}_p$ in $\vec{x}' = A\vec{x} + \vec{g}$, we get

$$\vec{v} = A\left[\vec{v}t + \vec{\eta}\right] + \vec{g}$$

$$\Rightarrow \quad \vec{v} = tA\vec{v} + A\vec{\eta} + \vec{g}$$

Equating similar coefficients on both sides:

$O(t)$: $\quad A\vec{v} = 0 \quad$ —— ①

Other terms: $\quad \cancel{A\vec{v}}\ A\vec{\eta} = \vec{v} - \vec{g} \quad$ — ②

from ①, $\quad A\vec{v} = 0 \quad \Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \vec{v} = 0$

$$\Rightarrow \left.\begin{aligned} -2v_1 + 2v_2 &= 0 \\ v_1 - v_2 &= 0 \end{aligned}\right\} v_1 = v_2$$

If $v_1 = \mu$ (some scalar), then $v_2 = \mu$

$$\therefore \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now, from ②: $\quad A\vec{\eta} = \vec{v} - \vec{g}$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \mu - 1 \\ \mu - 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \mu - 1 \\ \mu - 2 \end{bmatrix}$$

Row operation: Set Row 2 = $\frac{1}{2}$ Row 1 + Row 2

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \mu - 1 \\ \frac{1}{2}(\mu - 1) + (\mu - 2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \mu - 1 \\ \frac{3\mu}{2} - \frac{5}{2} \end{bmatrix}$$

We must set $\frac{3\mu}{2} - \frac{5}{2} = 0 \longrightarrow \mu = \frac{5}{3}$ for a solution.

Then $-2\eta_1 + 2\eta_2 = \frac{5}{3} - 1 = \frac{2}{3}$

$$\Rightarrow -\eta_1 + \eta_2 = \frac{1}{3}$$

Let $\eta_2 = K$, then $\eta_1 = \eta_2 - \frac{1}{3} = K - \frac{1}{3}$

$\downarrow$ arbitrary constant

$$\therefore \vec{\eta} = \begin{bmatrix} K - \frac{1}{3} \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$$

Without loss of generality, we can choose $K = 0$.

$$\therefore \vec{\eta} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$$

$$\therefore \vec{X}_p = \vec{v}t + \vec{\eta} = \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{X}_p = \frac{5}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}}$$

Ex! $\vec{g}$ involving exponential terms.

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x} + e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{———} \qquad (*)$$

- Homogeneous Solution:- $\vec{x}' = A\vec{x}, \qquad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

Put $\vec{x} = \vec{\xi} e^{rt} \Rightarrow \vec{\xi} r e^{rt} = A\vec{\xi} e^{rt}$

$$\Rightarrow (A - rI)\vec{\xi} = 0$$

Eigenvalues: $|A - rI| = 0 \rightarrow (-2-r)(-2-r) - 1 = 0$

$$\rightarrow (2+r)^2 = 1$$

$$\rightarrow 2+r = \pm 1$$

$$\rightarrow r_1 = -1, \quad r_2 = -3$$

Eigenvectors: (i) $(A - r_1 I)\vec{\xi}^{(1)} = 0 \rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Rightarrow \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(ii) $(A - r_2 I)\vec{\xi}^{(2)} = 0 \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Rightarrow \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Homogeneous Solution: $\vec{X}_H = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$

- Particular Solution:- Since $\vec{g}$ has a e^{2t} term in $(*)$, we have to try $\vec{X}_p = \vec{v} e^{2t}$.

$$\Rightarrow \vec{X}_p' = 2\vec{v} e^{2t}$$

$$\Rightarrow \quad 2\vec{v}e^{2t} = A\vec{v}e^{2t} + \vec{g}$$

$$\Rightarrow \quad 2\vec{v}e^{2t} = A\vec{v}\,e^{2t} + e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Cancelling the e^{2t} terms, we have

$$2\vec{v} = A\vec{v} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \quad A\vec{v} - 2\vec{v} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad (A - 2I)\,\vec{v} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\frac{\begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix}}{15}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{-1}{15}\begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/15 \\ 1/15 \end{bmatrix}$$

$$\Rightarrow \quad \vec{v} = \begin{bmatrix} 4/15 \\ 1/15 \end{bmatrix}$$

$\therefore$ Particular solution, $\vec{X}_p = \begin{bmatrix} 4/15 \\ 1/15 \end{bmatrix} e^{2t}$

$$\therefore \quad \vec{X} = \vec{X}_H + \vec{X}_p$$

$$= c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-3t} + \begin{bmatrix} 4/15 \\ 1/15 \end{bmatrix}e^{2t}$$

Ex: $\vec{g}$ with exponential terms, ~~where~~ $e^{\alpha t}$, where α is one of the eigenvalues.

$$\vec{X}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\vec{X} + e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now $\vec{g} = e^{-t}\begin{bmatrix}1\\0\end{bmatrix}$.

The homogeneous problem is the same as in the previous example.

$\Rightarrow \quad r_1 = -1 \quad \& \quad r_2 = -3$.

- Now if we try $\vec{X}_p = \vec{v}e^{-t}$, then

$$-\vec{v}e^{-t} = A\vec{v}e^{-t} + e^{-t}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \quad (A+I)\vec{v} = -\begin{bmatrix}1\\0\end{bmatrix}.$$

We cannot solve for $\vec{v}$ since $|A+I| = 0$

- Therefore, we try $\vec{X}_p = \vec{v}te^{-t} + \vec{\eta}e^{-t}$

Thus $\vec{X}_p' = \vec{v}e^{-t} - \vec{v}te^{-t} + \vec{\eta}(-e^{-t})$

$$\rightarrow \vec{v}e^{-t} - \vec{v}te^{-t} - \vec{\eta}e^{-t} = A\left[\vec{v}te^{-t} + \vec{\eta}e^{-t}\right] + e^{-t}\begin{bmatrix}1\\0\end{bmatrix}$$

Cancelling e^{-t} terms, we get

$$\rightarrow \vec{v} - \vec{v}t - \vec{\eta} = tA\vec{v} + A\vec{\eta} + \begin{bmatrix}1\\0\end{bmatrix}$$

Equating terms of similar type on both sides:

$O(t)$ terms: $A\vec{v} = -\vec{v} \rightarrow (A+I)\vec{v} = 0$

Other terms: $\vec{v} - \vec{\eta} = A\vec{\eta} + \begin{bmatrix}1\\0\end{bmatrix}$

- $(A+I)\vec{v} = 0 \longrightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\longrightarrow \left.\begin{array}{l} -v_1 + v_2 = 0 \\ v_1 - v_2 = 0 \end{array}\right\} v_1 = v_2$$

$\therefore$ If $v_1 = \mu$, then $v_2 = \mu$

$\therefore \vec{v} = \mu\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where μ is a scalar.

- $(A+I)\vec{\eta} = \vec{v} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= \mu\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mu-1 \\ \mu \end{bmatrix}$$

Solve for $\vec{\eta}$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \mu-1 \\ \mu \end{bmatrix}$

Row operation: Row 2 = Row 1 + Row 2

$$\longrightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \mu-1 \\ 2\mu-1 \end{bmatrix}$$

We conclude that $2\mu - 1 = 0$ is needed $\longrightarrow \mu = \frac{1}{2}$

Then $-\eta_1 + \eta_2 = \mu - 1 = \frac{1}{2} - 1 = \frac{-1}{2}$

$$\Rightarrow -\eta_1 + \eta_2 = \frac{-1}{2}$$

Let $\eta_2 = k \Rightarrow \eta_1 = k + \frac{1}{2}$

$$\therefore \vec{\eta} = \begin{bmatrix} k + 1/2 \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

Since K is arbitrary, we can set $K=0$

$\therefore \quad \vec{\eta} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$\therefore$ Particular solution is

$$\vec{X}_p = \vec{v}\, t e^{-t} + \vec{\eta}\, e^{-t}$$

$$= \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{-t}$$

$$\boxed{\vec{X}_p = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{-t}}$$

Nonhomogeneous term with trigonometric terms:

Ex: $\vec{X}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{X} + \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ —— ①

Solution: We can replace the $\cos t$ term with e^{it}

since $\cos t = Re\left[e^{it} \right]$.

Let $\vec{Y}' = A\vec{Y} + e^{it} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ —— ②

Now $\vec{X}_p$ (our solution) $= Re\left[\vec{Y}\right]$.

Let $\vec{Y} = \vec{V} e^{it}$. Substitute in ②, we get

$\Rightarrow \quad i\vec{V} e^{it} = A\vec{V} e^{it} + e^{it} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \quad i\vec{V} = A\vec{V} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow (A - i)\vec{V} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2-i & 1 \\ 1 & -2-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2-i & 1 \\ 1 & -2-i \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} -2-i & -1 \\ -1 & -2-i \end{bmatrix}}{(2+i)^2 - 1} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2+4i} \begin{bmatrix} -(2+i) & -1 \\ -1 & -(2+i) \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2+4i} \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

Multiplying & dividing by $2-4i$, we get

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{2-4i}{20} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} (1-2i)(2+i) \\ 1-2i \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 4-3i \\ 1-2i \end{bmatrix}$$

$$\vec{y} = \vec{v} e^{it} = \frac{1}{10} \begin{bmatrix} 4-3i \\ 1-2i \end{bmatrix} e^{it}$$

$$\therefore \vec{x}_p = Re[\vec{y}] = Re\left[\frac{1}{10} \begin{pmatrix} 4-3i \\ 1-2i \end{pmatrix} e^{it} \right]$$

$$= \frac{1}{10} \operatorname{Re}\left\{\left(\begin{bmatrix}4\\1\end{bmatrix} + i\begin{bmatrix}-3\\-2\end{bmatrix}\right)(\cos t + i\sin t)\right\}$$

$$= \frac{1}{10}\left[\begin{bmatrix}4\\1\end{bmatrix}\cos t - \begin{bmatrix}-3\\-2\end{bmatrix}\sin t\right]$$

$$\boxed{\vec{x}_p = \frac{1}{10}\begin{bmatrix}4\cos t + 3\sin t\\ \cos t + 2\sin t\end{bmatrix}}$$