GEOMETRICAL INTERPRETATION OF A DIFFERENTIAL EQUATION

Consider a first order differential equation of the form

$$y' = f(x,y) \quad \text{where} \quad y' = \frac{dy}{dx}$$

ANALYTICAL METHOD

Step (1) Write down the differential equation

$$y' = f(x,y)$$

Step 2: Obtain the solution $y = \phi(x)$

GEOMETRICAL METHOD

Equivalent to drawing the direction/slope field

Drawing Integral curves (curves which are always tangent to the direction field.

Theorem: $y = \phi(x)$ is the solution of a differential equation

$\Updownarrow$

is an integral curve.

It is

Procedure for drawing the direction field:-

Step 1: Write the differential equation in the form

$$y' = f(x,y)$$

Step 2: Make a rough grid in the x-y plane. The range for x & y should be decided by looking at the differential equation. Typically, choose y around the equilibrium solution ($y' = 0$) i.e. $f(x,y) = 0$.

Step 3: At every point on the grid, we draw little dashes having the slope at that point. To do this, ~~choose~~ fix y, and plot the dashes for all values of x. Continue this process till all points on the grid are marked with little dashes. This is the **Direction Field**.

Step 4: Draw integral curves, smooth curves tangent to the direction field.

Step 5: See if you can comment on the nature of the solution by observing the direction field and integral curves.

Example 1: **Falling Object:**

$$F = m\frac{dv}{dt}$$: Newton's – 2nd Law

v : velocity
t : time
m : mass

Forces on the object, $F = \underbrace{mg}_{\text{weight of the object}} - \underbrace{kv}_{\text{resistance or friction from air}}$

We get $m\dfrac{dv}{dt} = mg - kv$

In standard form, we get

$$\frac{dv}{dt} + \left(\frac{k}{m}\right)v = g$$

Let us put some numbers into this equation.

Let $m = 1 \text{ kg}$,

$g = 10 \frac{m}{sec^2}$

$K = 2 \frac{kg}{sec}$

NOTE: Always make sure that the differential equation is dimensionally consistent, ie; every term has the same dimensions. (units)

~~After dot~~ The differential equation becomes

$$\frac{dv}{dt} + 2v = 10$$

Step 1: Writing it in the form $y' = f(x,y)$, we have

$$\boxed{\frac{dv}{dt} = 10 - 2v}$$

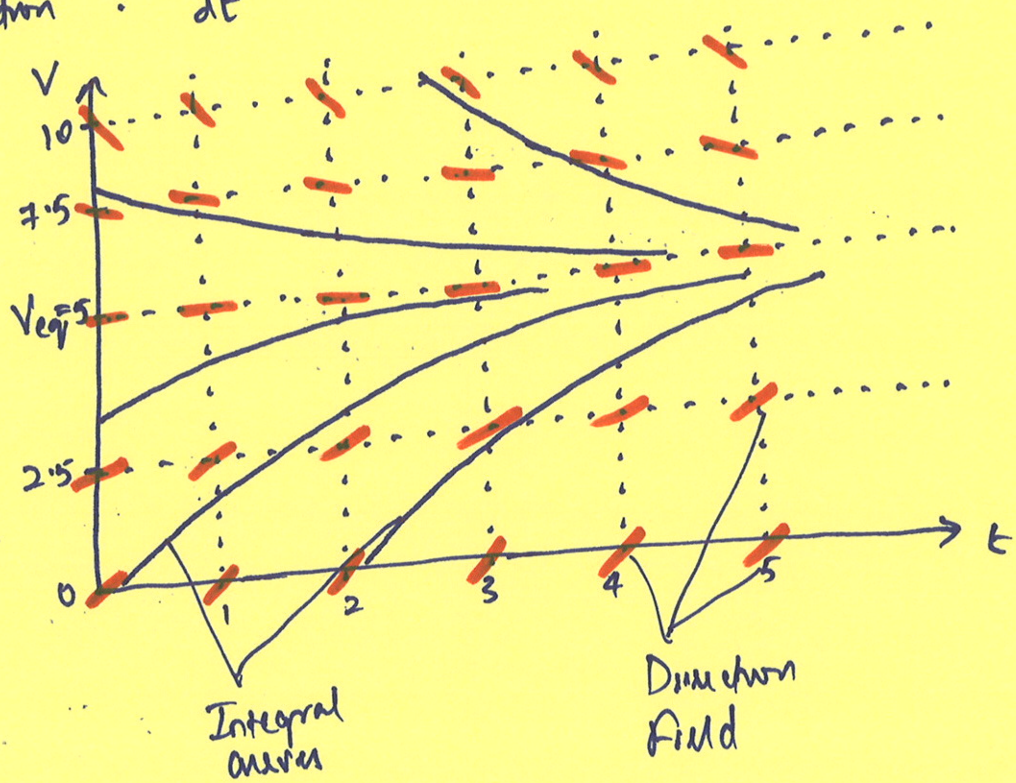
Step 2: We are trying to plot the direction field for the above equation. To choose the range for v, we first calculate equilibrium solution.

Equilibrium solution : $\frac{dv}{dt} = 0 \Rightarrow v_{eq} = 5$ m/sec.

Step 3: After the grid is drawn, mark the slopes.

for $v = 10$; $\frac{dv}{dt} = -10$

for $v = 0$; $\frac{dv}{dt} = 10$



What's the nature of the solution as $t \to \infty$?

Clearly, all integral curves approach $v = 5$. Therefore as $t \to \infty$, $v \to v_{eq} = 5$.

$v_{eq} = 5$ is therefore a STABLE EQUILIBRIUM.

Is there a more intelligent way to do this?

In the previous example, the right hand side does not involve 't' explicitly. Therefore, the slopes are constant for every v. But what if we have a more complicated equation?

Ex: $y' = \frac{-x}{y}$

Here the slopes change as x increases making our life difficult. But there is a better way (a HUMAN) way to draw integrals. Note that the previous procedure for drawing direction field is well suited for computers, as they are capable of ~~[illegible]~~ making innumerable calculations.

Alternate (Human) way of drawing direction fields and integral curves:-

Step 1: Write the differential equation in the form

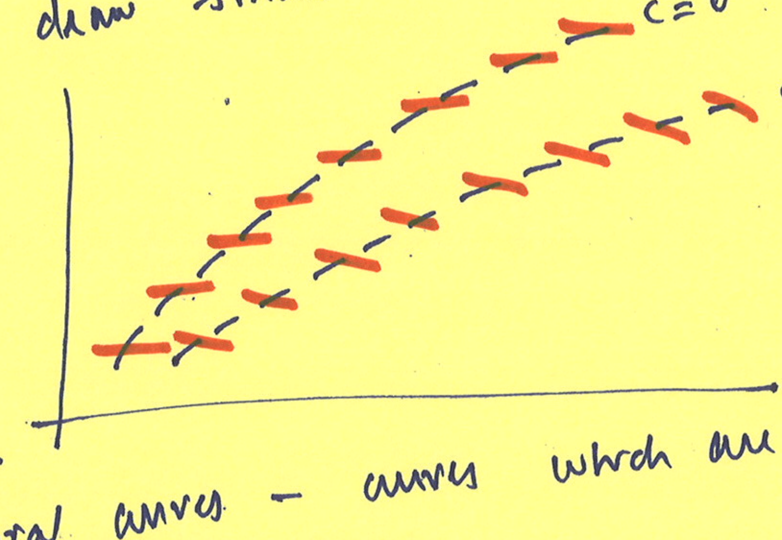
$$y' = f(x,y)$$

Step 2: Pick a slope C

Step 3: Since slope $= \frac{dy}{dx}$, we have the simpler algebraic equation, $f(x,y) = C$. Plot this equation. The curve so obtained is called an ISOCLINE.

(dashed lines)

Step 4: On the ISOCLINE, draw small line elements ~~of~~ with slope C. Continue by choosing a different C.

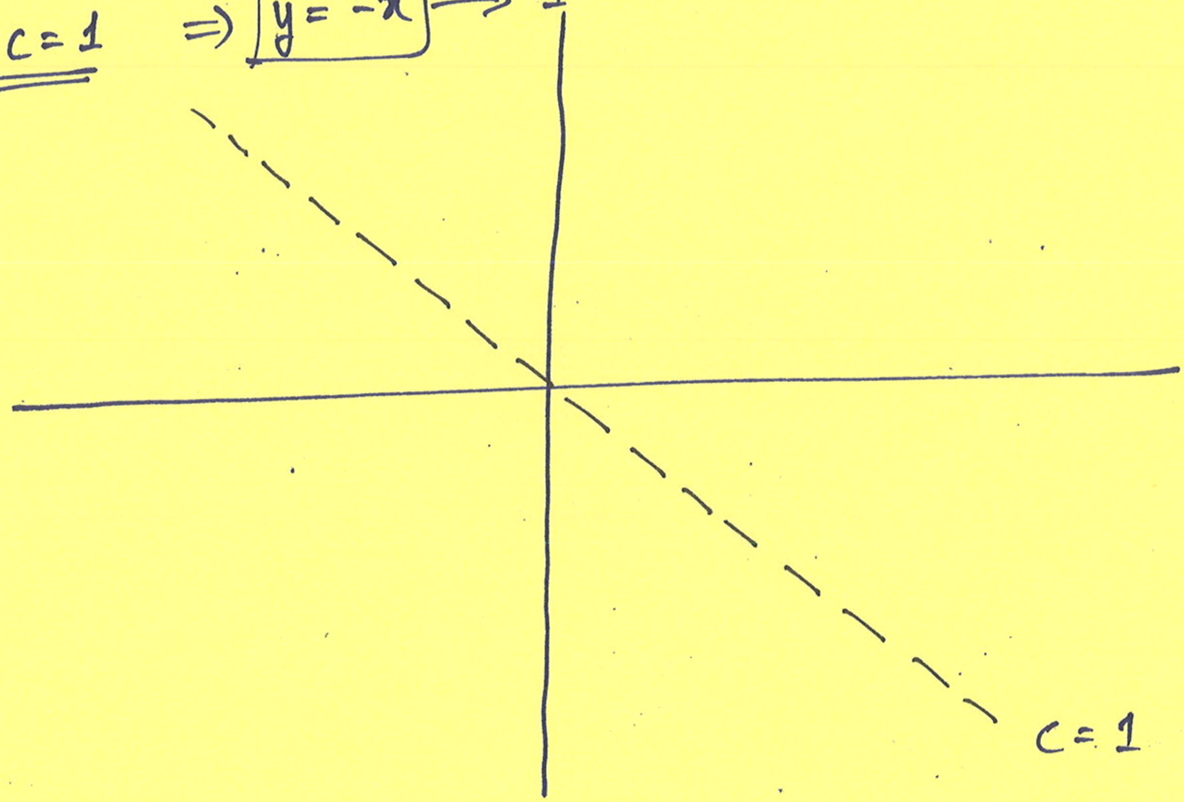


Step 5: Now we have the whole plane filled with little dashes. ~~[illegible]~~ Draw integral curves - curves which are tangent to the ~~dea~~ little dashes.

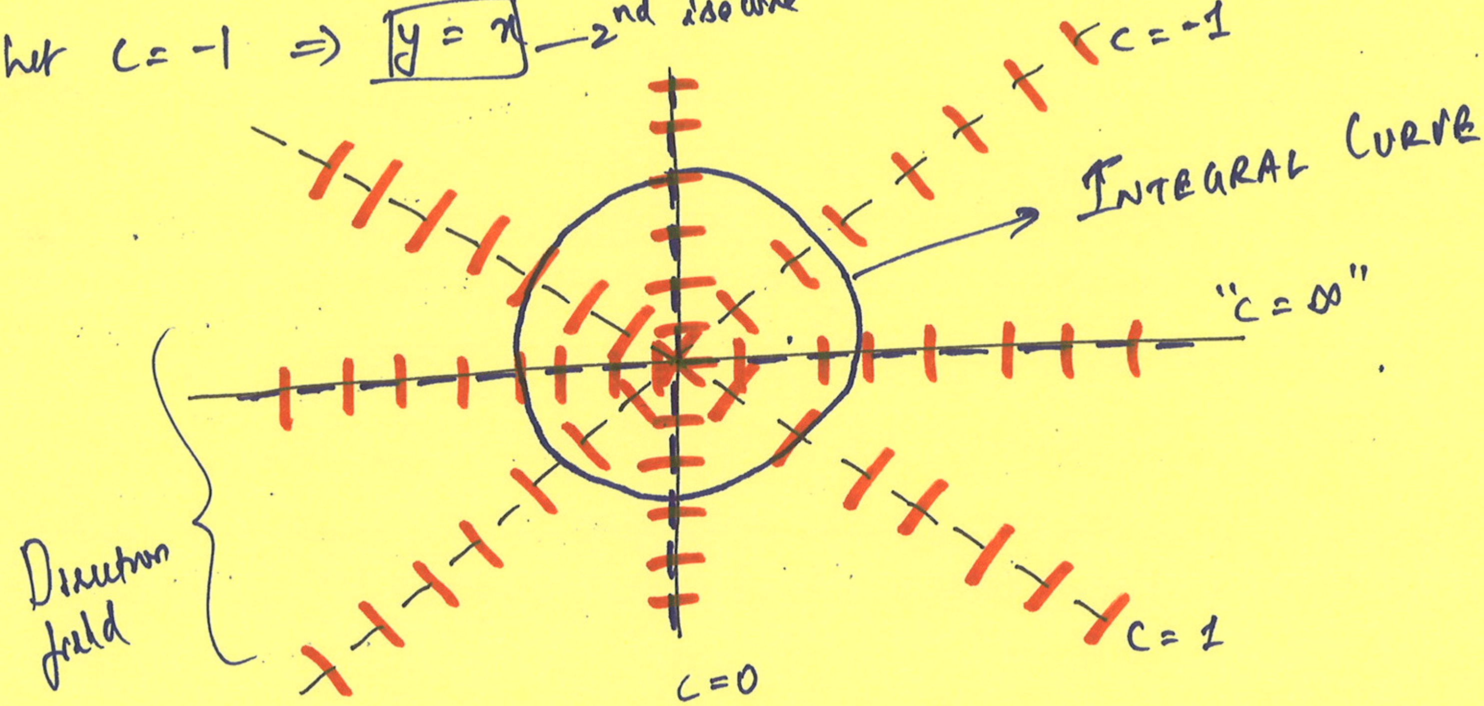
Ex: $y' = \frac{-x}{y}$.

Let $y' = c \Rightarrow \frac{-x}{y} = c \Rightarrow \boxed{y = \frac{-1}{c}x}$

Let $c = 1 \Rightarrow \boxed{y = -x} \rightarrow$ 1st isocline



Let $c = -1 \Rightarrow \boxed{y = x}$ — 2nd isocline



$$\left[\begin{array}{l} x = -cy \\ \Rightarrow x = 0 : y\text{-axis} \end{array}\right]$$

The integral curves are Circles, with centres at (0,0).

Can we verify this by solving the differential equation?

$$y' = -\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow y\, dy = -x\, dx$$

Integrating both sides

$$\int y\, dy = -\int x\, dx + C_1$$

$$\Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$\Rightarrow \boxed{x^2 + y^2 = 2C_1} \rightarrow \text{Circle with centre } (0,0).$$

The solution is indeed a circle, exactly as predicted by the direction field.