Solutions to Assignment-3

Problem 1: Solve the following differential equation using regular perturbation techniques for $\epsilon \ll 1$:

$$\frac{d^2y}{dx^2} - (1+\epsilon x)y = 0, \qquad y(0) = 1, \quad y'(0) = -1.$$

Obtain the solution upto $O(\epsilon)$ correction. Using any tool of your liking, obtain the exact or numerical solution of the above equation. Now make a plot of exact/numerical solution versus the approximate solution, i.e. $y_{\text{approx}} = y_0 + \epsilon y_1$, for different values of $\epsilon = 0.05, 0.1, 0.5$ in the range $x \in [0,2]$ and $y \in [0,1]$.

(A) $y'' - (1+\epsilon x)y = 0 \; ; \qquad y(0) = 1, \; y'(0) = -1$

Let $y = y_0 + \epsilon y_1 + \cdots$

$\Rightarrow (y_0'' + \epsilon y_1'' + \cdots) - (1+\epsilon x)(y_0 + \epsilon y_1 + \cdots) = 0$

with $y_0(0) + \epsilon y_1(0) + \cdots = 1$

$y_0'(0) + \epsilon y_1'(0) + \cdots = -1$

$\underline{\underline{O(1)}}$: $y_0'' - y_0 = 0$

with $y_0(0) = 1, \quad y_0'(0) = -1$

Using $y_0 = Ae^{\lambda t}$, we have

$Ae^{\lambda t}(\lambda^2 - 1) = 0 \Rightarrow \lambda = \pm 1$

$\therefore y_0(x) = c_1 e^{x} + c_2 e^{-x}$

At $x=0, y_0 = 1 \Rightarrow 1 = c_1 + c_2$

At $x=0, y_0' = -1 \Rightarrow -1 = c_1 - c_2$

$c_1 = 0, \; c_2 = 1$

$\therefore \boxed{y_0(x) = e^{-x}}$

$\underline{\underline{O(\epsilon)}}$: $y_1'' - x y_0 - y_1 = 0$

$\Rightarrow y_1'' - y_1 = x e^{-x}$

with $y_1(0) = 0, \; y_1'(0) = 0$

Homogeneous Solution:- $y_{1,H} = c_3 e^{-x} + c_4 e^{x}$

Particular Solution:- $y_{1,P} = Axe^{-x} + Bx^2 e^{-x}$

$y_{1,P}' = A\{e^{-x} - xe^{-x}\} + B\{e^{-x} 2x - x^2 e^{-x}\}$

$y_{1,P}'' = A\{-e^{-x} - e^{-x} + xe^{-x}\} + B\{2e^{-x} - 2xe^{-x}$

$$y''_{1,p} = A\{-e^{-x} - e^{-x} + xe^{-x}\} + B\{2e^{-x} - 2xe^{-x} - e^{-x}.2x + x^2e^{-x}\}$$

$$\therefore \left[A\{-2e^{-x} + xe^{-x}\} + B\{2e^{-x} - 4xe^{-x} + x^2e^{-x}\}\right] - \left[Axe^{-x} + Bx^2e^{-x}\right] = xe^{-x}$$

Comparing similar terms:

$\underline{\underline{e^{-x}}}$: $\quad -2A + 2B = 0 \Rightarrow B = A$

$\underline{\underline{xe^{-x}}}$: $\quad A - 4B - A = 1 \Rightarrow B = -1/4$

$\Rightarrow A = -1/4$

$\underline{\underline{x^2e^{-x}}}$: $\quad B - B = 0$ ✓

$$\therefore y_{1,p} = -\frac{1}{4}xe^{-x} - \frac{1}{4}x^2e^{-x}$$

$$\therefore y_1(x) = c_3e^{-x} + c_4e^{x} - \frac{1}{4}xe^{-x} - \frac{1}{4}x^2e^{-x}$$

$$y_1' = -c_3e^{-x} + c_4e^{x} - \frac{1}{4}\{e^{-x} - xe^{-x}\} - \frac{1}{4}\{2xe^{-x} - x^2e^{-x}\}$$

At $x=0$, $y_1=0 \Rightarrow c_3 + c_4 = 0$

At $x=0$, $y_1'=0 \Rightarrow -c_3 + c_4 - \frac{1}{4} = 0$

$$2c_4 - \frac{1}{4} = 0 \Rightarrow c_4 = \frac{1}{8}$$

$$\therefore c_3 = \frac{-1}{8}$$

$$\therefore y_1(x) = -\frac{1}{8}e^{-x} + \frac{1}{8}e^{x} - \frac{1}{4}xe^{-x} - \frac{1}{4}x^2e^{-x}$$

Problem 2: The equation of a pendulum of length l is written in non-dimensionless form as

$$\frac{d^2\theta}{dt^2} = -\sin\theta, \qquad \theta(0) = \phi, \quad \frac{d\theta}{dt}(t=0) = 0.$$

where time is non-dimensionalized by $\sqrt{l/g}$. Using regular perturbation techniques, obtain the solution for the case when $\phi \ll 1$. Compare this to the case of a simple pendulum when $\sin\theta$ is replaced by θ. Is your approximate solution uniformly valid in time?
Hint: Consider rescaling θ by exploiting the small parameter in the problem.

(A)

Given that $0 < \phi << 1$.

From the initial condition, it should clear that the angle θ varies periodically between $+\phi$ and $-\phi$.

To exploit the fact that $\phi << 1$, we first rescale θ as :

$$\boxed{\theta = \phi \cdot y} \quad \text{—— (2)}$$

where $y \sim \text{Ord}(1)$ and $\phi << 1$.

Substituting θ into the governing equation, we get

$$\frac{d^2(\phi y)}{dt^2} = -\sin(\phi y)$$

$$\Rightarrow \phi \cdot \frac{d^2 y}{dt^2} = -\left[y\phi - \frac{y^3\phi^3}{6} + \frac{y^5\phi^5}{120} + \cdots \right]$$

$$\Rightarrow \frac{d^2 y}{dt^2} + y - \phi^2 \cdot \frac{y^3}{6} + \phi^4 \cdot \frac{y^5}{120} + \cdots = 0$$

with $y(0) = 1$

and $\frac{dy}{dt}(\cancel{t}=0) = 0$ —— (3)

The governing equation for y involves the small parameter ϕ only in terms of ϕ^2. The correct small parameter is ϕ^2.

So let $y(t) = y_0(t) + \phi^2 y_1(t) + \phi^4 y_2(t) + \cdots$

... ordering terms according to ϕ^2, we get

The governing equation for y involves the small parameter ϕ only in terms of ϕ^2. The correct small parameter is ϕ^2.

So let $y(t) = y_0(t) + \phi^2 y_1(t) + \phi^4 y_2(t) + \dots$

Substituting and ordering terms according to ϕ^2, we get

$\underline{O(1)}$:

$$\left.\begin{aligned} &\frac{d^2 y_0}{dt^2} + y_0 = 0 \\ &y_0(0) = 1 \\ &\frac{dy_0}{dt}(0) = 0 \end{aligned}\right\} \quad y_0(t) = \cos t$$

$\underline{O(\phi^2)}$:

$$\frac{d^2 y_1}{dt^2} + y_1 = \frac{1}{6} y_0^3$$

$$y_1(0) = 0$$

$$\frac{dy_1}{dt}(0) = 0$$

Note

$$\cos^3 t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$$

$$\frac{d^2 y_1}{dt^2} + y_1 = \frac{1}{6}\cos^3 t$$

$$= \frac{1}{6}\left[\frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right]$$

Solving for y_1, we get

$$y_1(t) = \frac{1}{96}\sin t\,(\underbrace{6t}_{\text{secular term.}} + \sin 2t)$$

Problem 3: The equation of a projectile with a linear drag force is given by the equation

$$\frac{\partial^2 y}{\partial t^2} + \epsilon \frac{\partial y}{\partial t} + 1 = 0; \qquad y(0) = 0, \quad \frac{\partial y}{\partial t}(t = 0) = 1,$$

where $\epsilon > 0$ is the drag coefficient. Using perturbation theory, obtain an approximate solution in the limit of small drag, correct upto $O(\epsilon^2)$.

(A)

Projectile:

$$\frac{d^2y}{dt^2} = -\epsilon \frac{dy}{dt} - 1 \quad ; \quad y(0) = 0, \quad \frac{dy}{dt}(t=0) = 1$$

Let $y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$

Substituting, we get

$O(\epsilon^0)$: $\quad y_0'' + 1 = 0$

with $y_0(0) = 0$; $y_0'(0) = 1$

$O(\epsilon^1)$: $\quad y_1'' = -y_0'$

with $y_1(0) = 0$; $y_1'(0) = 0$

$O(\epsilon^2)$: $\quad y_2'' = -y_1'$

with $y_2(0) = y_2'(0) = 0$

Solving sequentially, we get

$$y_0(t) = t - \frac{t^2}{2}$$

$$y_1(t) = \frac{1}{6}(3t^2 - t^3)$$

$$y_2(t) = \frac{1}{24}(4t^3 - t^4)$$

and so on.

$\therefore y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$

$\therefore y(t) = \quad y_0(t) + \epsilon y_1(t)$

Problem 4: Solve the following differential equation using perturbation techniques:

$$\epsilon\frac{d^2y}{dx^2}+2\frac{dy}{dx}+e^x=0, \qquad y(0)=0, \quad y(1)=0.$$

Clearly determine where the boundary layer is located, obtain the scaling for the boundary layer width, obtain the inner and outer solutions and construct a uniformly valid approximation. Make a plot showing the three solutions - inner, outer, uniformly valid solutions. Does your solution agree well with the exact/numerical solution of the equation?

(A) $\quad \epsilon y'' + 2y' + e^x = 0$

Let $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$

$O(1)$: $\quad 2y_0' + e^x = 0 \implies y_0' = \frac{-e^x}{2} \implies \boxed{y_0 = \frac{-e^x}{2} + c}$

Let there be a boundary layer near $x = 0$:-

Let $x = \delta X$

$y(x) = Y(X)$

$\frac{dy}{dx} = \frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{1}{\delta}\frac{dY}{dX}$

$\frac{d^2y}{dx^2} = \frac{1}{\delta^2}\frac{d^2Y}{dX^2}$

$\frac{\epsilon}{\delta^2}\frac{d^2Y}{dX^2} + 2\cdot\frac{1}{\delta}\frac{dY}{dX} + e^{\delta X} = 0$

$\implies \frac{\epsilon}{\delta}\frac{d^2Y}{dX^2} + 2\frac{dY}{dX} + \delta e^{\delta X} = 0$

Choosing $\delta = \epsilon$:-

$\frac{d^2Y}{dX^2} + 2\frac{dY}{dX} + \epsilon e^{\epsilon X} = 0$

Let $Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots$

$\underline{O(1)}$: $\dfrac{d^2Y_0}{dX^2} + 2\dfrac{dY_0}{dX} = 0$

$$\frac{dY_0}{dX} + 2Y_0 = C_1$$

$$Y_0(X) = -\frac{1}{2}\cancel{C_1}e^{-2X} \quad 0 = C_2 - \frac{1}{2}C_1 + C_2 \Rightarrow \Big| \; C_2 = \frac{C_1}{2}$$

Using $Y_0(X=0) = 0$, $0 = -\frac{1}{2}C_1 e^0 + C_2$

$$\Rightarrow Y_0(X) = -\frac{1}{2}C_1 e^{-2X} + \frac{C_1}{2}$$

$$\boxed{Y_0(X) = \frac{C_1}{2}\left(1 - e^{-2X}\right)}$$

Now using $y_0(x=1) = 0$, we get

$$y_0(x=1) = -\frac{e^1}{2} + C = 0 \Rightarrow C = \frac{e}{2}$$

$$\therefore y_0(x) = -\frac{e^x}{2} + \frac{e}{2}$$

$\underline{\text{Matching}}$: $\lim_{x\to 0} y_0(x) = \lim_{X\to\infty} Y_0(X) = y_{overlap}$

$$\Rightarrow -\frac{1}{2} + \frac{e}{2} = \frac{C_1}{2} \Rightarrow C_1 = e-1$$

Note that $y_{overlap} = \frac{C_1}{2} = \frac{e-1}{2}$

$$\therefore Y_0(X) = \left(\frac{e-1}{2}\right)\left(1 - e^{-2X}\right)$$

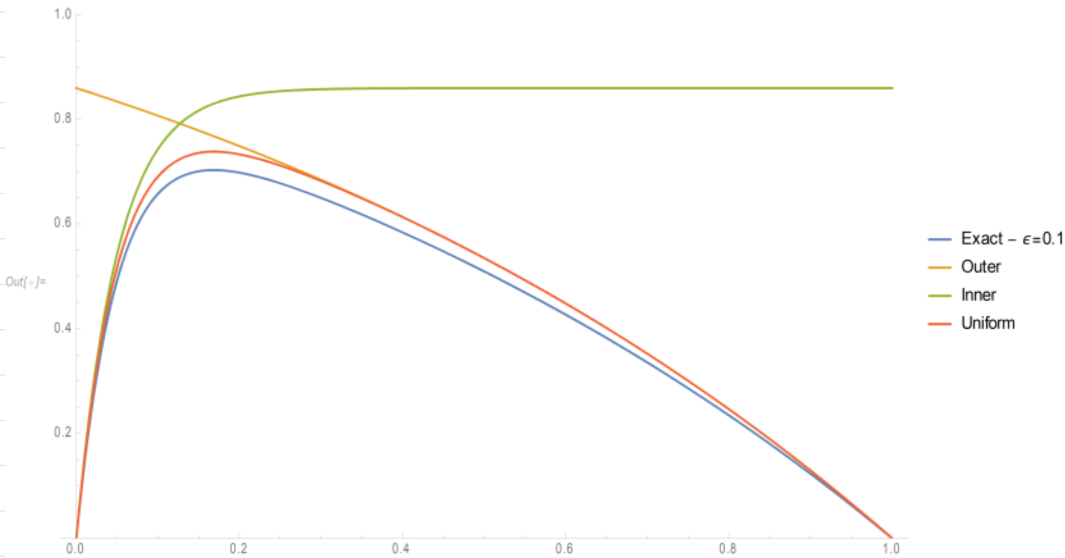
$\underline{\text{Uniform Soln}}$: $y^{(0)}_{unif}(x) = y_0(x) + Y_0(X) - y_{overlap}$

$$= \frac{e}{2} - \frac{e^x}{2} + \left(\frac{e-1}{2}\right)\left(1 - e^{-2X}\right) - \frac{e-1}{2}$$

$$= \frac{e}{2} - \frac{e^x}{2} + \cancel{\left(\frac{e-1}{2}\right)} - \left(\frac{e-1}{2}\right)e^{-\frac{2x}{\epsilon}} - \cancel{\left(\frac{e-1}{2}\right)}$$

$$= \frac{e}{2} - \frac{e^x}{2} + \cancel{\left(\frac{e-1}{2}\right)} - \left(\frac{e-1}{2}\right) e^{-2x/\epsilon} - \cancel{\left(\frac{e-1}{2}\right)}$$

$$= \frac{e}{2} - \frac{e^x}{2} - \left(\frac{e-1}{2}\right) e^{-2x/\epsilon}$$



Problem 5: Solve the following differential equation using perturbation techniques for $\epsilon \ll 1$:

$$\epsilon \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} - y = 0, \qquad y(0) = 1, \quad y(1) = 1.$$

This problem is a bit more tricky than problem-1. First obtain the outer solution and examine the regions nearly the two boundaries closely. After determining the inner solutions, obtain a uniformly valid solution.

Make a plot in Matlab or Mathematica comparing the exact/numerical solution with the perturbation solution.

Let $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$

Substituting

$$\epsilon \left[y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + \cdots \right] - x^2 \left[y_0' + \epsilon y_1' + \epsilon^2 y_2' + \cdots \right] - \left[y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots \right] = 0$$

$O(1)$: $\quad -x^2 y_0' - y_0 = 0$

$$\Rightarrow \boxed{y_0 = C_0 e^{1/x}}$$

If $x = 0$, then $\frac{1}{x} \to \infty$ & hence $y_0 \to \infty$. Let us examine the two limits $x = 0$ & $x = 1$ for a boundary layer.

Putting boundary layer at $x = 1$:-

Let $x = 1 - \delta X \Rightarrow \frac{dx}{dX} = -\delta$

and $y(x) = Y(X)$

If $x=0$, then $\frac{1}{x}\to\infty$ & hence $y_0\to\infty$. Let us examine the two limits $x=0$ & $x=1$ for a boundary layer.

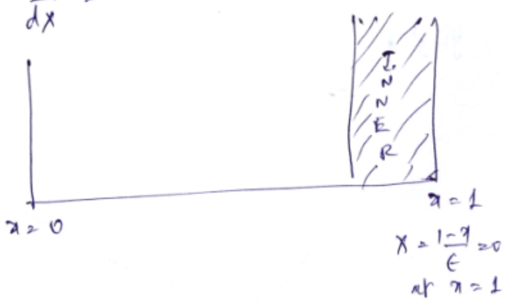
<u>Putting boundary layer at $x=1$:-</u>

Let $x = 1-\delta X \Rightarrow \frac{dx}{dX} = -\delta$

and $y(x) = Y(X)$

$$\frac{dy}{dx} = \frac{-1}{\delta}\frac{dY}{dX}$$

& $$\frac{d^2y}{dx^2} = \frac{+1}{\delta^2}\frac{d^2Y}{dX^2}$$



(Going from Inner to Outer $\overleftarrow{X\to\infty}$)

Substituting, we have

$$\frac{\epsilon}{\delta^2}Y'' + (1-\delta X)^2\frac{1}{\delta}Y' - Y = 0$$

$$\Rightarrow \frac{\epsilon}{\delta}Y'' + (1-\delta X)^2 Y' - \delta Y = 0$$

natural choice for δ is $\boxed{\delta=\epsilon} \Rightarrow X = \frac{1-x}{\epsilon}$

$$\Rightarrow Y'' + (1-\epsilon X)^2 Y' - \epsilon Y = 0$$

Let $Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots$

$\underline{O(1)}$: $Y_0'' + Y_0' = 0$

with $Y_0(X=0) = 1$

$\Rightarrow Y_0(X) = C_1 + C_2 e^{-X}$ with $Y_0(0) = 1$.

$\Rightarrow 1 = C_1 + C_2 \Rightarrow C_2 = 1 - C_1$

$\therefore Y_0(X) = C_1 + (1-C_1)e^{-X}$: This solution decay as $X\to\infty$ and is acceptable

The constant C_1 is unknown and has to be determined by matching. (iv)

Clearly, having a boundary layer at $x=1$ is acceptable. But since $y_0(x)\to\infty$ as $x\to 0$, we cannot have $y_0(x) = C_0 e^{1/x}$ as the outer solution.

We therefore require another boundary layer at $x=0$, to satisfy the condition $y(0) = 1$. We use another boundary layer scale Z such that

$x = Z\cdot\beta$ where $Z \sim \text{ord}(1)$ & $\beta \ll 1$

boundary layer scale ~

$x = Z \cdot \beta$ where $Z \sim \text{Ord}(1)$ &

$\beta << 1$

& $y(x) = Y_{left}(Z)$

Substituting $x = Z\beta$, we have

$$\frac{\epsilon}{\beta^2}\frac{d^2 Y_{left}}{dZ^2} - Z^2\beta^2 \cdot \frac{1}{\beta}\frac{dY_{left}}{dZ} - Y_{left} = 0$$

$$\Rightarrow \frac{\epsilon}{\beta^2}\frac{d^2 Y_{left}}{dZ^2} - \beta Z^2 \frac{dY_{left}}{dZ} - Y_{left} = 0$$

There are two possibilities :

$\frac{\epsilon}{\beta^2} \sim \beta$ such that $\beta \sim \epsilon^{1/3}$

or $\frac{\epsilon}{\beta^2} \sim 1$ such that $\beta \sim \epsilon^{1/2}$

In the first case, the third term dominates and since $Y_{left} \sim \text{Ord}(1)$, this cannot be balanced.

The only consistent boundary layer for the inner solution on the left boundary is $\boxed{\beta = \epsilon^{1/2}.}$

With this choice, we have

$$\frac{d^2 Y_{left}}{dz^2} - Y_{left} = \epsilon^{1/2} z^2 \frac{dY_{left}}{dz}$$

Let $Y_{left} = Y_0 + \epsilon^{1/2} Y_1 + \cdots$

Leading order $O(1)$:-

$$\frac{d^2 Y_{0,left}}{dz^2} - Y_{0,left} = 0$$

$$\Rightarrow \quad Y_{0,left}(z) = C_3 e^{z} + C_4 e^{-z}$$

With $Y_{0,left}(0) = 1$, we have $C_3 + C_4 = 1$

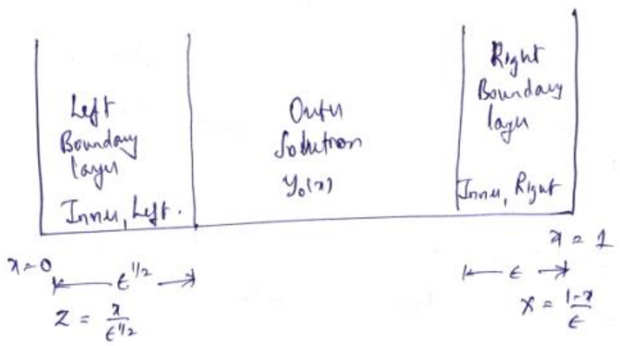
As $z \to \infty$, $e^{z} \to \infty$. To suppress this large growth, we require $C_3 = 0$.

Matching also requires that $C_0 = 0$ since $y_0(x) \to \infty$ as $x \to 0$.

Therefore, we have $C_0 = 0$, and hence outer solution is

$$y_0(x) = 0$$

General picture :-



Matching the right boundary layer to the outer solution :- (ii)

$$\lim_{x \to 1} y_0(x) = \lim_{X \to \infty} Y_{right}.$$

where $y_0(x) = C_0 e^{1/x} = 0$ since $C_0 = 0$

& $Y_{right} = C_1 + (1 - C_1) e^{-X}$

$$\Rightarrow \quad 0 = C_1 \quad \Rightarrow \quad \boxed{C_1 = 0}$$

Now we have all the necessary constants except C_4. In this ... two boundary layers, one on left & ...

Now we have all the necessary constants ... problem, there are two boundary layers, one on left & one on right, & an outer solution, $y_0(x)$, in the middle.

$$Y_{0,left} = C_4 e^{-Z}$$

$$Y_{0,right} = e^{-X}$$

$$y_0(x) = 0 \quad \text{(outer solution)}$$

~~Matching the left boundary layer to the outer solution.~~

$$\cancel{\lim_{x\to 0} y_{outer} = \lim_{Z\to\infty} Y_{left}}$$

$$\cancel{\to 0 =}$$

Since $Y_{0,left}(Z=0) = 1$, we have $C_4 = 1$.

Uniform or Composite Solution :-

$$y_{unif} = y_0(x) + Y_{0,left}(Z) + Y_{0,right}(X) - y_{left,match} - y_{right,match}$$

$$= 0 + e^{-\left(\frac{1-x}{\epsilon}\right)} + e^{-x/\epsilon^{1/2}} - 0 - 0$$

$$\boxed{y_{unif.} = e^{\frac{x-1}{\epsilon}} + e^{-x/\epsilon^{1/2}}}$$