ME7100: Advanced Topics in Mathematical Tools
Assignment-2

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Due date: 6th February 2022, before the class begins. Make plots on a separate sheet of paper and include them along with your solutions.

Problem 1: Solve the following differential equation using regular perturbation techniques for $\epsilon \ll 1$:

$$\frac{d^2y}{dx^2} - (1 + \epsilon x)y = 0, \qquad y(0) = 1, \quad y'(0) = -1.$$

Obtain the solution upto $O(\epsilon)$ correction. Using any tool of your liking, obtain the exact or numerical solution of the above equation. Now make a plot of exact/numerical solution versus the approximate solution, i.e. $y_{\text{approx}} = y_0 + \epsilon y_1$, for different values of $\epsilon = 0.05, 0.1, 0.5$ in the range $x \in [0, 2]$ and $y \in [0, 1]$.

Problem 2: The equation of a pendulum of length l is written in non-dimensionless form as

$$\frac{d^2\theta}{dt^2} = -\sin\theta, \qquad \theta(0) = \phi, \quad \frac{d\theta}{dt}(t = 0) = 0.$$

where time is non-dimensionalized by $\sqrt{l/g}$. Using regular perturbation techniques, obtain the solution for the case when $\phi \ll 1$. Compare this to the case of a simple pendulum when $\sin\theta$ is replaced by θ. Is your approximate solution uniformly valid in time?
Hint: Consider rescaling θ by exploiting the small parameter in the problem.

Problem 3: The equation of a projectile with a linear drag force is given by the equation

$$\frac{\partial^2 y}{\partial t^2} + \epsilon\frac{\partial y}{\partial t} + 1 = 0; \qquad y(0) = 0, \quad \frac{\partial y}{\partial t}(t = 0) = 1,$$

where $\epsilon > 0$ is the drag coefficient. Using perturbation theory, obtain an approximate solution in the limit of small drag, correct upto $O(\epsilon^2)$.

Problem 4: Solve the following differential equation using perturbation techniques:

$$\epsilon\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + e^x = 0, \qquad y(0) = 0, \quad y(1) = 0.$$

Clearly determine where the boundary layer is located, obtain the scaling for the boundary layer width, obtain the inner and outer solutions and construct a uniformly valid approximation. Make a plot showing the three solutions - inner, outer, uniformly valid solutions. Does your solution agree well with the exact/numerical solution of the equation?

Problem 5: Solve the following differential equation using perturbation techniques for $\epsilon \ll 1$:

$$\epsilon\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - y = 0, \qquad y(0) = 1, \quad y(1) = 1.$$

This problem is a bit more tricky than problem-1. First obtain the outer solution and examine the regions nearly the two boundaries closely. After determining the inner solutions, obtain a uniformly valid solution.

Make a plot in Matlab or Mathematica comparing the exact/numerical solution with the perturbation solution.