Kinematics - 1

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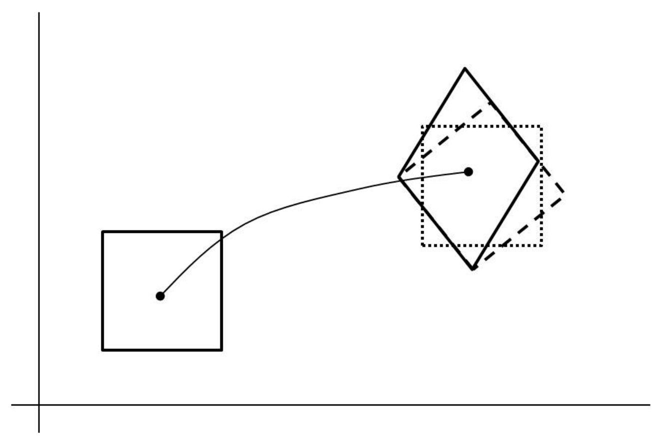
1 Kinematics of local fluid motion

Fluids are different from solids in one important way: fluids cannot resist tangential (shear) stresses but solids do. Solids respond by undergoing deformation (strains) which develop internal stresses, eventually resisting the external loading on them. For most solids, at small strains, the stress is proportional to strain, the Hooke's law. But in the case of fluids, they continue to deform as long as the applied shear stress is present. The internal mechanism to develop internal stresses similar to solids is absent. Hence fluids, at least the simple ones, cannot resist strains. But different fluids respond by the rate at which they undergo this continuous deformation. Thus we must relate stress to rates of deformations, not deformation itself.

Kinematics deals with formulating an exact mathematical description to describe this rate of deformation. The elementary motions can be classified into following categories:

- translation
- solid-body rotation
- straining
  1. extensional strain
  2. shear strain



There are two different methods to describe fluid flows:

1. Lagrangian viewpoint
2. Eulerian viewpoint

The two viewpoints only differ in the choice of independent variables.

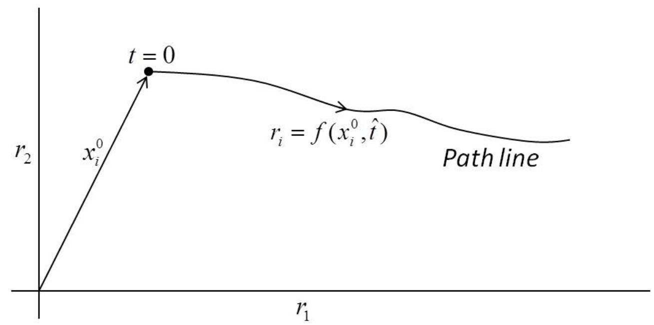
	Independent variables
Lagrangian	x_i^0 and $\hat{t}$ Initial point
Eulerian	x_i and t Fixed point in space

2 Lagrangian viewpoint

The Lagrangian viewpoint is an natural extension of particle mechanics. Each particle is identified by its initial position, x_i^0 at time $\hat{t} = 0$. To measure a physical property in a flow, we could leave a small probe which will make measurements as it traverses the fluid. For simplicity, let us assume that the mass of the probe is negligible so that the path line taken by the probe is a faithful depiction of the local flow at its position. The use of weather or sounding balloons to measure quantities such as atmospheric pressure, temperature, humidity, etc. is an example of a Lagrangian measurement.

If this were to be a temperature probe, then the temperature in Lagrangian variables is then given by

$$T = T_L\left(x_i^0, \hat{t}\right). \tag{1}$$



The subscript L denotes that the temperature is measured in a Lagrangian framework. The variation of temperature in time and space would depend on where this probe get carried along with the flow as time passes. Let r_i be the position of a material point, the probe in this case, at time $\hat{t}$, i.e.

$$r_i = f\left(x_i^0, \hat{t}\right) \tag{2}$$

describes the path taken by the material point as it travels in space. It is evident that knowing the function f is key to the success of any Lagrangian measurement. The velocity and acceleration of a particle are then given by

$$v_i = \frac{\partial f}{\partial \hat{t}}; \qquad a_i = \frac{\partial^2 f}{\partial \hat{t}^2}. \tag{3}$$

If there are multiple material points, then v_i and a_i will in general will be different for each of these points. In steady flow, the initial position will completely determine the path taken by the material point at all times since $f\left(x_i^0, \hat{t}\right)$ does not change with time.

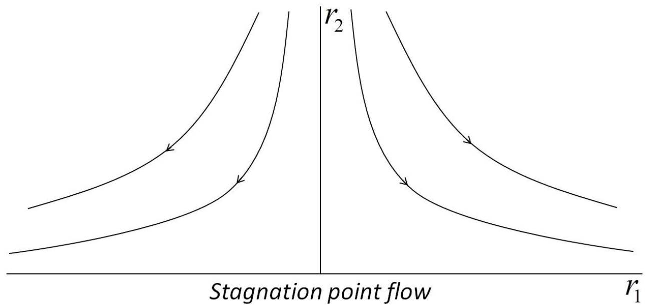
2.1 Stagnation point flow

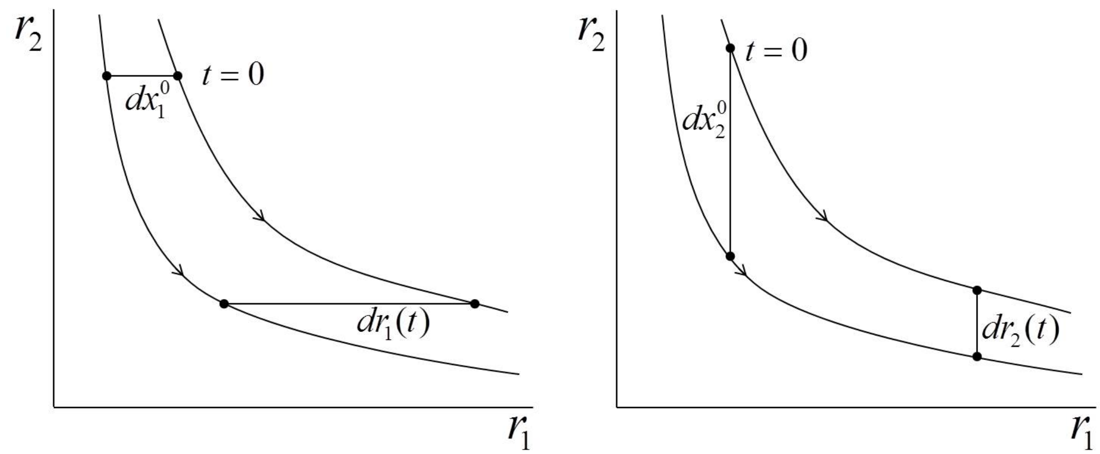
Let us examine the above concepts with the help of a common flow. If we release tracer particles from the point $\left(x_1^0, x_2^0, x_3^0\right)$ at $\hat{t} = 0$, then the particle trajectory in this flow is given by

$$r_1 = f(x_1^0, \hat{t}) = x_1^0 \exp(c\hat{t}), \tag{4}$$
$$r_2 = f(x_2^0, \hat{t}) = x_2^0 \exp(-c\hat{t}), \tag{5}$$
$$r_3 = x_3^0. \tag{6}$$

The velocity along the trajectory of the material point is given by

$$v_1 = \frac{\partial}{\partial \hat{t}} f(x_1^0, \hat{t}) = c x_1^0 \exp(c\hat{t}), \tag{7}$$
$$v_2 = \frac{\partial}{\partial \hat{t}} f(x_2^0, \hat{t}) = -c x_2^0 \exp(-c\hat{t}), \tag{8}$$
$$v_1 = \frac{\partial}{\partial \hat{t}} f(x_3^0, \hat{t}) = 0. \tag{9}$$





(a) (b)

Eliminating $\hat{t}$, we have

$$r_1 r_2 = x_1^0 x_2^0,$$
$$r_2 = \frac{x_1^0 x_2^0}{r_1}; \qquad r_3 = x_3^0.$$

The path made by the particle in the $r_1 - r_2$ plane is a rectangular hyperbola. The distance between any two particles is given by

$$dr_1 = \frac{\partial f}{\partial x_1^0} dx_1^0 + \frac{\partial f}{\partial x_2^0} dx_2^0, \tag{10}$$
$$= dx_1^0 \exp(c\hat{t}), \tag{11}$$
$$dr_2 = dx_2^0 \exp(-c\hat{t}). \tag{12}$$

In this flow, two points along a horizontal (vertical) line continue to remain on the horizontal (vertical) line at all times.

It should be clear from the above example that in the Lagrangian viewpoint, the major dependent variable is the position vector, r_i, and not the velocity.

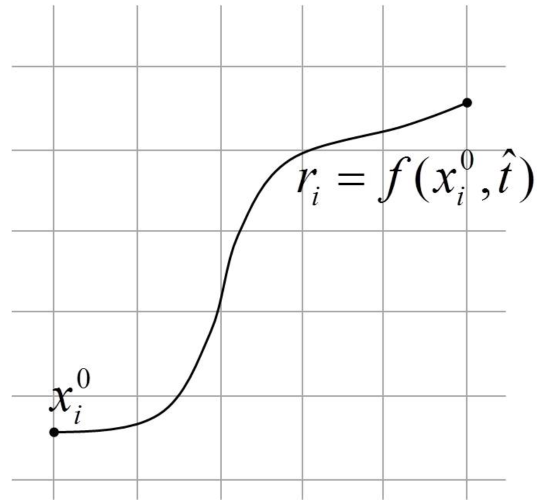
3 Eulerian viewpoint

In this viewpoint, we watch a fixed point, x_i, in space as time t proceeds. All flow properties such as r_i, v_i, etc. are functions of x_i and t. In the context of previous discussion, a temperature measurement in an Eulerian view is like keeping the temperature probe fixed at a single point in space and measuring the temperature 'field' at a single point in space. In this kind of measurement, to determine the temperature field, a large number of fixed probes are necessary. Fixed weather stations on the surface of earth is an example of an Eulerian measurement.

The temperature measured from many of fixed probes at locations x_i and time t, i.e., $T_E(x_i, t)$, tells us how the temperature changes in space. At a fixed point, $T_E(x_i, t)$ give us the local history of temperature.

The particle position vector in Eulerian variables is simply

$$r_i = r_i(x_i, t) = x_i. \tag{13}$$



To draw the equivalence between Lagrangian and Eulerian viewpoints, all is require is to look at the same point in space and at the same time. In an Eulerian viewpoint, the point is simply x_i and in the Lagrangian viewpoint, we require a material point to pass through this point. Hence

$$r_i = r_i(x_i, t) = x_i \quad \text{and} \quad t = \hat{t}. \tag{14}$$

Substituting $r_i = x_i$ into the Lagrangian definition, we have $x_i = f(x_i^0, t)$.

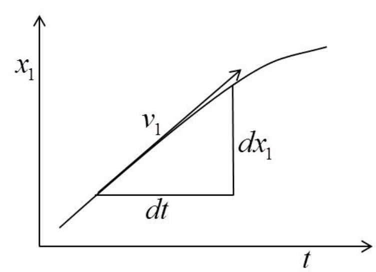
Now x_i is the position vector in Eulerian coordinates and $f(x_i^0, t)$ tells us the history of a particle that started off at x_i^0 when $t = 0$ and is now at x_i after a time t.

4 Streamlines

In order to compare the path taken by a material point in a Lagrangian viewpoint to something similar in an Eulerian viewpoint, we need to define a *streamline*. A streamline is an imaginary line that at any instant is tangent to the local velocity vector.

If dx_i is a differential element along a streamline, the tangency condition is expressible by three equations:

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}. \tag{15}$$



The above relations can also be written as

$$\epsilon_{ijk} v_j dx_k = 0 \quad \text{or} \quad \mathbf{v} \times d\mathbf{x} = 0. \tag{16}$$

A unique direction to streamline is determined at all points in space except at those points at which the velocity goes to zero. If the velocity is zero at some point, it is possible for two or more streamlines to exist at that point.

4.1 Stagnation point flow

Let us revisit the stagnation point flow that we encountered earlier. The Eulerian-Lagrangian transformation is given by

$$x_1 = x_1^0 \exp(ct), \quad x_2 = x_2^0 \exp(-ct). \tag{17}$$

The velocity 'field' is found by substituting x_i^0 into v_i:

$$v_1 = cx_1^0 \exp(ct) = cx_1, \tag{18}$$
$$v_1 = -cx_2^0 \exp(-ct) = -cx_2. \tag{19}$$

Therefore,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2, \tag{20}$$
$$= cx_1\mathbf{e}_1 - cx_2\mathbf{e}_2. \tag{21}$$

Clearly $\mathbf{v}$ is independent of time. Hence this is a steady flow. The streamlines are obtained by using the relation:

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2},$$

thus

$$\frac{dx_2}{dx_1} = \frac{v_2}{v_1} = \frac{-cx_2}{cx_1} = -\frac{x_2}{x_1}.$$

Integrating this equation, we get

$$x_2 = \frac{A}{x_1}. \tag{22}$$

So in this case, streamlines coincides with pathlines. In general, pathlines and streamlines are identical in a steady flow.
It is important to keep in mind that a flow which appears steady in one coordinate system may appear unsteady in another coordinate system.

5 Substantial or Material derivative

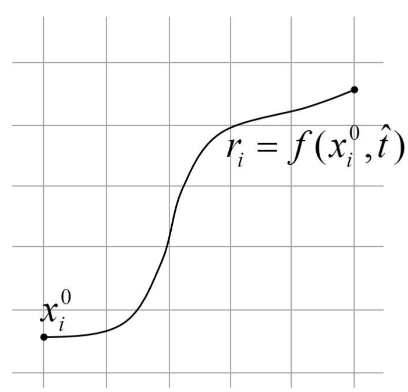
In this Eulerian viewpoint, we lose the ability to track individual fluid particles. Moreover to describe the time-rate of change of a quantity, we need to know its history in the Lagrangian framework. The *substantial* or *material* derivative is an expression that allows us to formulate, in Eulerian variables, a time derivative as we follow a material particle.

Let $\mathcal{H}$[1] be any arbitrary property of the flow under consideration. The parameter $\mathcal{H}$ may be expressed in Lagrangian variables as $\mathcal{H}_L(x_i^0, \hat{t})$ and in Eulerian variables as $\mathcal{H}_E(x_i, t)$. For an equivalence between the two frameworks, we require

$$\mathcal{H} = \mathcal{H}_L(x_i^0, \hat{t}) = \mathcal{H}_E(x_i, t). \tag{23}$$

This equality makes sense only if we substitute the Eulerian-Lagrangian transformation into the above equation:

$$\mathcal{H} = \mathcal{H}_L(x_i^0, \hat{t}) = \mathcal{H}_E\left(x_i = f(x_i^0, \hat{t}), t = \hat{t}\right). \tag{24}$$



The rate of change of $\mathcal{H}$ as we follow a particle if found from the chain rule:

$$\frac{\partial \mathcal{H}}{\partial \hat{t}} = \frac{\partial \mathcal{H}_L}{\partial \hat{t}}, \tag{25}$$
$$= \frac{\partial \mathcal{H}_E}{\partial x_i}\frac{\partial x_i}{\partial \hat{t}} + \frac{\partial \mathcal{H}_E}{\partial t}\frac{\partial t}{\partial \hat{t}}, \tag{26}$$
$$= \frac{\partial \mathcal{H}}{\partial x_i}\frac{\partial f}{\partial \hat{t}} + \frac{\partial \mathcal{H}}{\partial t}. \tag{27}$$

[1]The alphabet $\mathcal{H}$ is in honor of Prof. Bud Homsy.

But since $v_i = \frac{\partial f}{\partial \hat{t}}$, we have

$$\underbrace{\frac{\partial \mathcal{H}_L}{\partial \hat{t}}}_{\text{Lagrangian variables}} = \underbrace{\frac{\partial \mathcal{H}_E}{\partial t} + v_i \frac{\partial \mathcal{H}_E}{\partial x_i}}_{\text{Eulerian variables}} . \tag{28}$$

The combination on the RHS has the physical interpretation of a time derivative following a fluid particle. This substantial derivative occurs so frequently that Stokes gave it a special symbol:

$$\boxed{\frac{\partial ()}{\partial \hat{t}} \equiv \frac{D()}{Dt} \equiv \frac{\partial ()}{\partial t} + v_i \frac{\partial ()}{\partial x_i}}. \tag{29}$$

In symbolic notation,

$$\boxed{\frac{D(\cdot)}{Dt} \equiv \frac{\partial (\cdot)}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})(\cdot)}. \tag{30}$$

The above boxed equations are one of the most important expressions that you will encounter in this course. The first term in RHS defines the local rate of change. It vanishes unless there is a change with time at a fixed location. The second term in RHS is the convective derivative. It vanishes unless there are spatial gradients in the flow. This gradient is advected or convected with a flow velocity v_i.

Example-1:
Let us examine the above analysis by taking $\mathcal{H}$ to be the position vector, r_j. Since $r_j = r_j(x_i, t) = x_i$ in Eulerian coordinates, we have

$$\begin{aligned} \frac{Dr_j}{Dt} &= \frac{\partial r_j}{\partial t} + v_i \frac{\partial r_j}{\partial x_i}, \\ &= 0 + v_i \delta_{ij}, \\ &= v_j. \end{aligned} \tag{31}$$

This example just shows that the change of displacement is equal to the velocity.

Example-2:
The expression for substantial derivative can also be used to derive boundary conditions in a free surface flow. If $\eta(\mathbf{x}, t)$=constant describes a material surface in a fluid, then η as a quantity is invariant for all fluid elements on the surface. Hence

$$\frac{D\eta}{Dt} = 0.$$

Let $\eta = z - f(x, y, t)$. Hence setting the substantial derivative to zero gives us

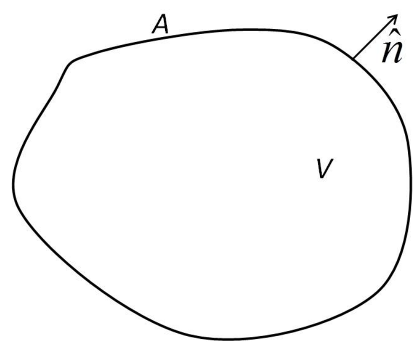
$$\begin{aligned} & \frac{\partial \eta}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\eta = 0, \\ & \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} + w\frac{\partial \eta}{\partial z} = 0, \\ \Longrightarrow \quad & w = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}. \end{aligned}$$

Later in the course, we will see that the above expression is used to obtain the kinematic condition at a free surface.

6 Conservation of mass

The requirement of conservation of mass imposes certain restrictions on the velocity field, and although these are not strictly 'kinematical', it is convenient to consider them at this stage.

Consider a closed surface A whose position is fixed relative to some coordinate axes, and it encloses a volume V. If ρ is the density of the fluid at a position $\mathbf{x}$ and time t, then the mass of the enclosed fluid is $\int \rho dV$ and the net rate at which mass is flowing outwards is $\int \rho \mathbf{u} \cdot \mathbf{n} dA$.



In the absence of any sources or sinks, mass is conserved.

$$\Longrightarrow \frac{d}{dt}\left\{\int \rho dV\right\} = -\int \rho \mathbf{u} \cdot \mathbf{n} dA, \tag{32}$$

$$\Longrightarrow \int \frac{\partial \rho}{\partial t} dV = -\int \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) dV, \tag{33}$$

$$\Longrightarrow \int \left\{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u})\right\} dV = 0. \tag{34}$$

Since the choice of V is arbitrary, we require

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \tag{35}$$

This is one of the most fundamental equations of fluid mechanics. Expanding the divergence, we get

$$\underbrace{\frac{1}{\rho}\frac{D\rho}{Dt}}_{\text{fractional change in density}} + \underbrace{\boldsymbol{\nabla} \cdot \mathbf{u}}_{\text{fractional change in volume}} = 0. \tag{36}$$

The first term represents the fractional change in density whereas the second term represents the fractional change in volume. It is easy to verify the latter statement.

The volume τ of a material body of a fluid changes as a result of movement of each element $\mathbf{n}ds$ of the bounding surface:

$$\frac{d\tau}{dt} = \int \mathbf{u} \cdot \mathbf{n} ds, \tag{37}$$

$$= \int \boldsymbol{\nabla} \cdot \mathbf{u} \, d\tau. \tag{38}$$

Rate at which the volume of a material element changes is given by

$$\underbrace{\lim_{\tau \to 0} \frac{1}{\tau}\frac{d\tau}{dt}}_{\text{rate of dilation}} = \lim_{\tau \to 0} \frac{1}{\tau} \int \boldsymbol{\nabla} \cdot \mathbf{u} \, d\tau = \underbrace{\boldsymbol{\nabla} \cdot \mathbf{u}}_{\text{rate of expansion}}. \tag{39}$$

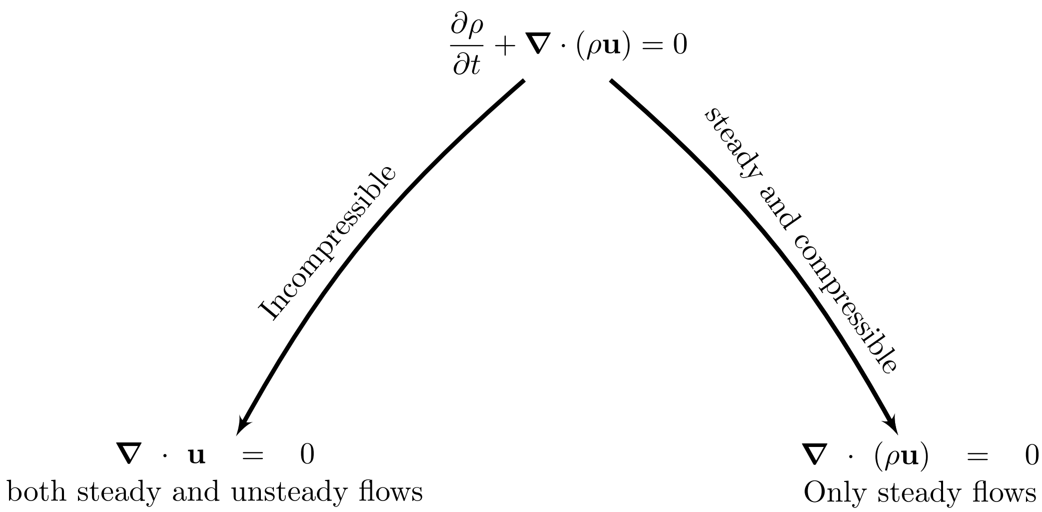
A fluid is said to be incompressible when the density of an element of fluid is not affected by changes in pressure. Thus, for an incompressible fluid, the rate of change of ρ following the motion is zero, i.e,

$$\text{Incompressible fluid:} \quad \frac{D\rho}{Dt} = 0. \tag{40}$$

In this case, the mass conservation equation becomes

$$\text{Incompressible fluid:} \quad \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{41}$$

In summary, two possibilities emerge from the same mass conservation equation:



6.1 Use of stream function to satisfy mass conservation

If flow is either 2D or axisymmetric, then $\boldsymbol{\nabla} \cdot \mathbf{u}$ or $\boldsymbol{\nabla} \cdot (\rho \mathbf{u})$ is the sum of only two terms. Then, mass conservation can be used to define a scalar field, ψ.

Ex: Consider $\mathbf{u} = (u, v, 0)$ and u, v are independent of z. Then

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \quad \Longrightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{42}$$

Define $u = \frac{\partial \psi}{\partial y}$ and $v - \frac{\partial \psi}{\partial x}$, then it is easy to verify that the continuity equation is identically satisfied. We therefore have

$$\delta\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy, \tag{43}$$

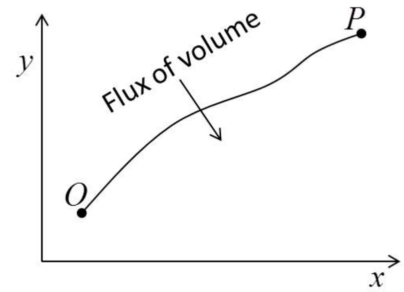
$$= u dy - v dx. \tag{44}$$

This is an exact differential. If ψ_0 is the value of ψ at a point 0, then ψ at any point can be obtained by integrating the above equation:

$$\psi - \psi_0 = \int u dy - v dx. \tag{45}$$

The integral on the RHS is a line integral along any arbitrary curve connecting 0 and P. Physically, the RHS is the flux of fluid volume across the curve OP. The flux is taken to be positive in the anti-clockwise sense about P.

The flux of volume across the closed curve from any two different paths joining O and P is zero when the flow is incompressible. The flux represented by the integral is therefore independent of the choice of the path between 0 and P.
Since the flux of volume across any curve joining two points is equal to the difference between values of ψ at these two points, ψ *is therefore constant along a streamline*. Hence we can assign ψ a name, the *stream-function*.



ψ can also be thought of as a "vector potential" such that

$$\mathbf{u} = \boldsymbol{\nabla} \times \mathbf{B} \qquad \text{where} \qquad \mathbf{B} = (0, 0, \psi). \tag{46}$$

Let $\psi_1 - \psi_2 = \epsilon$, the average velocity q between the two streamlines ψ_1=constant and ψ_2=constant is given by the

$$q \approx \frac{\epsilon}{\text{(distance between the streamlines)}}. \tag{47}$$

So if the two streamlines come close to each other, then q increases. Such a situation arises when we have flow over an aerofoil. Right above the airfoil, the streamlines get bunched together (but never touching each other) and hence the fluid velocity increases above the airfoil. We will later see that this results in a decrease in pressure above the airfoil, and hence a lift force, as a consequence of Bernoulli's equation.