

ME5310: Incompressible Fluid Flow Assignment - 1

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Office hours with TAs: Wednesday, 5:30 PM-7:00 PM, discussion room, KRC building,

Due date: 26th August 2024, before the class begins.

Problem 1

If the second order tensor $\mathbf{T}$ is defined as

$$T_{ij} = aS_{ij} + bP_{ijk}q_k + c_k d_i S_{kj} + f\delta_{ij},$$

where a , b and f are scalars, $\mathbf{c}$, $\mathbf{d}$ are vectors, $\mathbf{S}$ is an arbitrary rank-2 tensor, δ is the Kronecker delta and P_{ijk} is an arbitrary rank-3 tensor.

The summation for the three indices $\{i, j, k\}$ goes from 1 to 3. Write down the expressions for the following tensor components:

- (i) T_{11}
- (ii) T_{12}
- (iii) Contraction of T_{ij} : Make $i = j$ to obtain an expression for T_{ii} . Note that T_{ii} is equal to $\text{trace}(\mathbf{T})$.

Problem 2

Using index notation, prove the following vector algebra identities between the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$:

- (i) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$.
- (ii) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{z})(\mathbf{v} \cdot \mathbf{w})$: Binet-Cauchy identity.
- (iii) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{z}$
- (iv) $[(\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w})] + (\mathbf{v} \cdot \mathbf{w})^2 = v^2 w^2$, where v and w are the magnitudes of $\mathbf{v}$ and $\mathbf{w}$ respectively.

This identity is referred to as the Lagrange's identity* named after the great Italian mathematician Joseph-Louis Lagrange. The Lagrangian frame of reference is named after the same Lagrange.

Problem 3

Using index notation, prove the following vector calculus identities:

*Some extra fundaes: If we have a set of ordered pairs $\{a_1, a_2, a_3, \dots\}$ and $\{b_1, b_2, b_3, \dots\}$, then Lagrange's identity, as it appears in algebra, can also be written in the form

$$\left(\sum_{k=1}^n a_k^2\right) + \left(\sum_{k=1}^n b_k^2\right) - \left(\sum_{k=1}^n a_k b_k\right) = \left(\sum_{i=1}^{n-1} \sum_{j=i+1}^n (a_i b_j - a_j b_i)^2\right).$$

The LHS is simply a combination of magnitudes of vectors $\mathbf{a}$, $\mathbf{b}$ and scalar product $\mathbf{a} \cdot \mathbf{b}$, and RHS involves components of $\mathbf{a} \times \mathbf{b}$. For $n = 2$, this identity was first proved by Diophantus of Alexandria, considered the 'father of algebra' in 3rd century AD and later by the great Indian mathematician & astronomer Brahmagupta, in 7th century AD. Brahmagupta was also the first person to formulate rules to use zero in algebra. Now, the special case of $n = 2$ is referred to as Brahmagupta-Fibonacci identity or simply the Fibonacci identity. Notice that Lagrange's identity is a special case of Binet-Cauchy identity.

(i) $\nabla \cdot \phi \mathbf{v} = \nabla \phi \cdot \mathbf{v} + \phi(\nabla \cdot \mathbf{u})$ where ϕ is a scalar and $\mathbf{v}$ is a vector.

(ii) $\nabla \cdot \nabla \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$. LHS is simple $\nabla^2 \mathbf{u}$, the vector Laplacian.

(iii) $\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - [\mathbf{u} \times (\nabla \times \mathbf{u})]$

(iv) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}$

(v) $\nabla(\mathbf{v} \cdot \mathbf{w}) = [(\nabla \mathbf{v}) \cdot \mathbf{w}] + [(\nabla \mathbf{w}) \cdot \mathbf{v}]$ where $\mathbf{v}$ and $\mathbf{w}$ are vectors.

Hint: In some cases, it may be useful to show that starting from RHS, we get LHS. I leave it to your good judgement to decide where you want to use this hint.

Problem 4

If $\mathbf{A}$ is symmetrical and $\mathbf{B}$ is anti-symmetrical, show that $\mathbf{A} : \mathbf{B} = 0$.

Problem 5

Prove that if T is a symmetric tensor (of rank-2),

$$\mathbf{T} : \nabla \mathbf{v} = \nabla \cdot (\mathbf{T} \cdot \mathbf{v}) - \mathbf{v} \cdot (\nabla \cdot \mathbf{T})$$

where $\mathbf{v}$ is a vector.

Problem 6

If $\mathbf{r}$ is a position vector with scalar components (x_1, x_2, x_3) , show that

1. $(\nabla \cdot \mathbf{r}) = 3$

2. $(\nabla \times \mathbf{r}) = 0$

3. $[\mathbf{r} \times (\nabla \cdot \mathbf{v} \mathbf{v})] = [\nabla \cdot \mathbf{v}(\mathbf{r} \times \mathbf{v})]$ where $\mathbf{v}$ is a function of position.

Problem 7

Write down the below equation using index notation, i.e. in cartesian form:

$$\frac{\partial}{\partial t} \rho \mathbf{u} + \nabla \cdot \rho \mathbf{u} \mathbf{u} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g}$$

where $\mathbf{u}$ is a vector, $\boldsymbol{\tau}$ is a rank-2 tensor and $\mathbf{g}$ is acceleration due to gravity.

Cauchy, another great French mathematician, is well known for a wide range of contributions. More importantly for us, he extended the Euler's equations of inviscid fluids to more general class of fluids. This is referred to as the Cauchy momentum equation (written in vector form with a stress tensor σ and external forcing $\mathbf{f}$):

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = \nabla \cdot \sigma + \mathbf{f}$$

Incidentally, if we use a simpler form of the stress tensor by assuming it to be linearly proportional to the gradient of velocity with p as the pressure, i.e., if $\sigma = -p\mathbf{I} + \mu \nabla \mathbf{u}$, then the Cauchy momentum equation reduces to the Navier-Stokes equation. We refer to the proportionality constant, μ , as viscosity.