Stability of superposed fluids:-

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$$\omega^2 + a\omega + b = 0 \qquad \eta = \eta_0 e^{\omega t + ikx}$$

where
$$a = 2ik\frac{(\rho_1 U_1 + \rho_2 U_2)}{\rho_1 + \rho_2}$$

$$b = -\frac{k^2(\rho_2 U_2^2 + \rho_1 U_1^2)}{\rho_1 + \rho_2} - \frac{(\rho_2 - \rho_1)}{(\rho_1 + \rho_2)}gk + \frac{k^3\sigma}{(\rho_1 + \rho_2)}$$

Since $e^{\omega t} = e^{(\omega_r + i\omega_i)t} = e^{\omega_r t} \cdot e^{i\omega_i t}$

If $Re(\omega) > 0$, $e^{\omega_r t} \to \infty$ as $t \to \infty$

$\Rightarrow$ Flow (system) is unstable

If $Re(\omega) < 0$, $e^{\omega_r t} \to 0$ as $t \to \infty$

$\Rightarrow$ System is stable.

If $Re(\omega) = 0$, $e^{\omega_r t} = 1$ for all time.

$\Rightarrow$ Systems oscillates forever.

Special Cases:-

(i) $U_1 = U_2 = 0$

$$a = 0$$
$$b = -\frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)}gk + \frac{k^3\sigma}{(\rho_2 + \rho_1)}$$

$$\omega^2 + b = 0$$

$$\Rightarrow \omega^2 = \left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)gk - \frac{k^3\sigma}{(\rho_2 + \rho_1)}$$

$$\omega = \pm\sqrt{\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)gk - \frac{k^3\sigma}{(\rho_2 + \rho_1)}}$$

Rayleigh-Taylor instability:-

If $\rho_2 > \rho_1$

ρ_2

ρ_1

instability occurs only if

$$\frac{(\rho_2-\rho_1)}{\cancel{(\rho_2+\rho_1)}} g \cancel{k} > \frac{k^{\cancel{3}\,2}\sigma}{\cancel{(\rho_2+\rho_1)}}$$

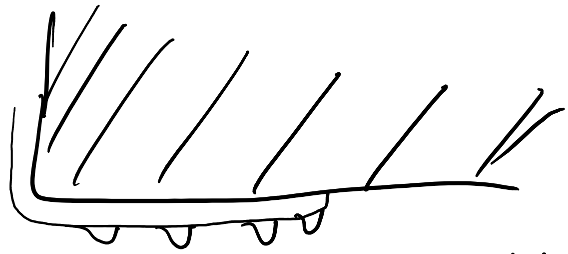
$$(\rho_2-\rho_1)g > k^2\sigma$$

or $$k^2 < \frac{(\rho_2-\rho_1)g}{\sigma}$$

$$\Rightarrow \quad k < \sqrt{\frac{(\rho_2-\rho_1)g}{\sigma}}$$

$$\therefore \lambda = \frac{2\pi}{k} > \underbrace{\sqrt{\frac{\sigma}{(\rho_2-\rho_1)g}}}_{l_c}$$

Instability occurs when $\lambda > l_c$

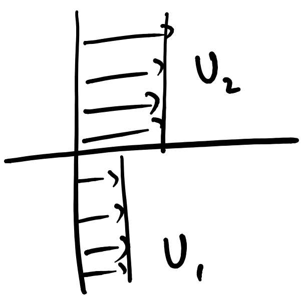


Surface tension stabilizes Rayleigh-Taylor instability at small scales (ie; for large k).

Kelvin-Helmholtz Instability:-

(i) $\rho_1 = \rho_2 = \rho_0$

$\omega^2 + a\omega + b = 0$ where



$a = ik(U_1+U_2)$

$-k^2(U_1^2+U_2^2) + \frac{k^3\sigma}{2\rho_0}$

$$\omega = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Instability occurs when

$$a^2 - 4b > 0 \implies -k^2(U_1+U_2)^2 - 4\left\{\frac{-k^2}{2}(U_1^2+U_2^2) + \frac{k^3\sigma}{2\rho_0}\right\} > 0$$

$$\implies -k^2(U_1^2+U_2^2+2U_1U_2) + 2k^2(U_1^2+U_2^2) - \frac{2k^3\sigma}{\rho_0} > 0$$

$$\implies k^2(U_1^2+U_2^2-2U_1U_2) - \frac{2k^3\sigma}{\rho_0} > 0$$

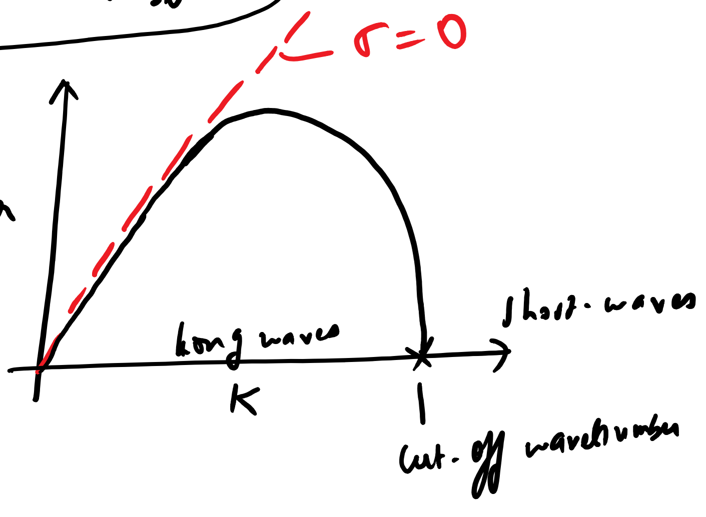
$$\implies k^2(U_1-U_2)^2 - \frac{2k^3\sigma}{\rho_0} > 0$$

$$\implies (U_1-U_2)^2 - \frac{2k\sigma}{\rho_0} > 0$$

$$\implies (U_1-U_2)^2 > \frac{2k\sigma}{\rho_0}$$

or $\boxed{|U_1-U_2| > \sqrt{\frac{2k\sigma}{\rho_0}}}$

For fixed U_1, U_2, ρ_0, σ, we get ω_r vs $k \implies$



$$k_{cut\text{-}off} = \frac{(U_1-U_2)^2\rho_0}{2\sigma}$$

$$\lambda_{cut\text{-}off} = \frac{2\pi \cdot 2\sigma}{(U_1-U_2)^2\rho_0}$$

for $k > k_{cut\text{-}off}$, no instability $\implies$ short-waves are stabilized by surface tension.

If $\sigma = 0$, instability occurs for all wavelengths.

Ocean surface:-

$\rho_2 = \rho_a \qquad U_2 = U$

Also ignore σ.

$\rho_1 = \rho_w = 1000 \qquad V_1 = 0$

$$\omega^2 + a\,\omega + b = 0$$

$$\eta = \eta_0 e^{\omega t + ikx}$$

where
$$a = 2ik\frac{(\cancel{\rho_1 U_1}^{\,0} + \rho_2 U_2)}{\rho_1 + \rho_2}$$

$$b = -\frac{k^2(\rho_2 U_2^2 + \cancel{\rho_1 U_1^2}^{\,0})}{\rho_1 + \rho_2} - \frac{(\rho_2 - \rho_1)}{(\rho_1 + \rho_2)}gk + \cancel{\frac{k^3 \sigma}{(\rho_1 + \rho_2)}}^{\,0}$$

$$a \approx \frac{2ik\,\rho_a U}{(\rho_w + \rho_a)} \approx \frac{2ik\,\rho_a U}{\rho_w}$$

$$b \approx -k^2 \frac{\rho_a U^2}{\rho_w} + gk$$

Instability when $a^2 - 4b > 0$

$$-\frac{4k^2 \rho_a^2 U^2}{\rho_w^2} - 4\left\{-\frac{k^2 \rho_a U^2}{\rho_w} + gk\right\} > 0$$

$$U^2\left\{-4\frac{k^2 \rho_a^2}{\rho_w^2} + \frac{4k^2 \rho_a}{\rho_w}\right\} - 4gk > 0$$

Ocean Surface:-

Ocean Surface:-

$$\rho_a = 1\ \mathrm{kg/m^3}\ ,\quad U_a = U \qquad 2$$

$$\rho_w = 1000\ \mathrm{kg/m^3}\ ,\quad U_w = 0 \qquad 1$$

$$\omega^2 + a\omega + b = 0 \qquad \omega = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

where

$$a = \frac{2ik(\cancelto{0}{\rho_1 U_1} + \rho_2 U_2)}{\rho_1 + \rho_2}$$

$$b = -\frac{k^2(\rho_2 U_2^2 + \cancelto{0}{\rho_1 U_1^2})}{\rho_1 + \rho_2} - \frac{(\rho_2 - \rho_1)}{(\rho_1 + \rho_2)} gk + \frac{k^3 \sigma}{(\rho_1 + \rho_2)}$$

$$a = 2ik\left(\frac{\rho_a}{\rho_w + \rho_a}\right) U$$

$$b = \frac{-k^2 \rho_a U^2}{\rho_a + \rho_w} - \frac{(\rho_a - \rho_w)}{(\rho_a + \rho_w)} gk + \frac{k^3 \sigma}{(\rho_a + \rho_w)}$$

Instability occurs when $a^2 - 4b > 0$

$$\Rightarrow \underbrace{-4k^2\left(\frac{\rho_a}{\rho_w + \rho_a}\right)^2 U^2 - 4\left\{\frac{-k^2 \rho_a U^2}{(\rho_a + \rho_w)} + \frac{(\rho_w - \rho_a)}{(\rho_w + \rho_a)} gk + \frac{k^3 \sigma}{(\rho_a + \rho_w)}\right\}}_{f(U,\, k,\, \underline{G})} > 0$$

$$\underline{G} = \{\rho_a, \rho_w, g, \sigma\}$$

$$f(U, k, G) = 0$$

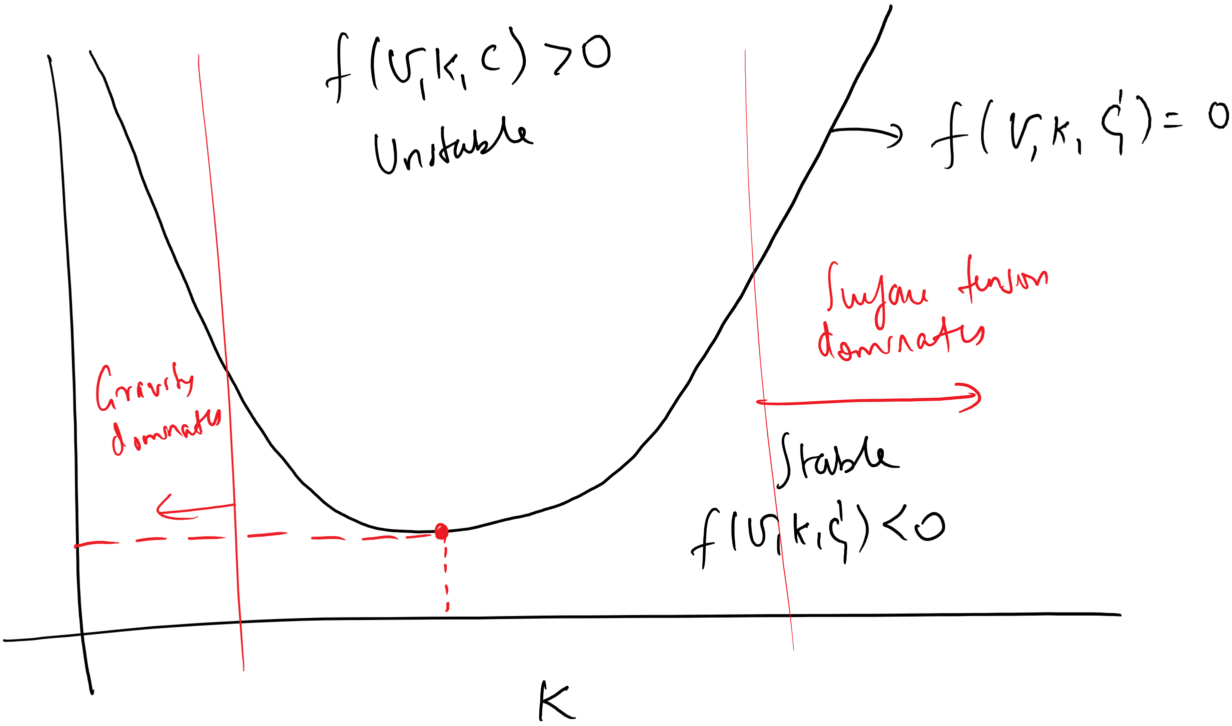
$$\Rightarrow U^2 \left\{ -4k^2 \left(\frac{\rho_a}{\rho_w + \rho_a}\right) \left[\frac{\rho_a}{\rho_a + \rho_w} - 1\right] \right\} - \frac{4(\rho_w - \rho_a)}{(\rho_w + \rho_a)} gk - \frac{4k^3 \sigma}{\rho_w + \rho_a} = 0$$

$$\Rightarrow U^2 \left\{ -4k^2 \left(\frac{\rho_a}{\rho_w+\rho_a}\right) \left[\frac{\rho_a}{\rho_w+\rho_a} - 1\right] \right\} - \frac{4(\rho_w-\rho_a)}{(\rho_w+\rho_a)} gk - \frac{\quad}{\rho_w+\rho_a}$$

$$U^2 \times -4k^2 \frac{\rho_a}{(\rho_w+\rho_a)} \left(\frac{-\rho_w}{\rho_w+\rho_a}\right) = \frac{4(\rho_w-\rho_a)}{(\rho_w+\rho_a)} gk + \frac{4\sigma}{(\rho_w+\rho_a)} k^3$$

$$\Rightarrow U^2 \times \frac{4k^2 \rho_a \rho_w}{(\rho_w+\rho_a)^2} = \frac{4(\rho_w-\rho_a)}{\cancel{(\rho_w+\rho_a)}} gk + \frac{4\sigma}{\cancel{(\rho_w+\rho_a)}} k^3$$

$$\Rightarrow \boxed{U^2 = \frac{(\rho_w^2-\rho_a^2)}{\rho_a \rho_w} \frac{g}{k} + \sigma \frac{(\rho_w+\rho_a)}{(\rho_a \rho_w)} k}$$



Minimum wind speed for instability :-

$$2U \frac{\partial U}{\partial k} = \frac{(\rho_w^2-\rho_a^2)}{\rho_w \rho_a} \cdot \left(\frac{-g}{k^2}\right) + \sigma \frac{(\rho_w+\rho_a)}{\rho_w \rho_a} = 0$$

$$\sigma \frac{\cancel{(\rho_w+\rho_a)}}{\rho\,\rho} = \frac{(\rho_w-\rho_a)\cancel{(\rho_w+\rho_a)}}{\rho\,\rho} \cdot \frac{g}{k^2}$$

$$\sigma \cancel{\frac{(\rho_w + \rho_a)}{\rho_w \rho_a}} = \frac{(\rho_w - \rho_a)\cancel{(\rho_w + \rho_a)}}{\cancel{\rho_w \rho_a}} \cdot \frac{g}{k^2}$$

$$\Rightarrow \quad k^2 = \frac{(\rho_w - \rho_a)\, g}{\sigma}$$

Using $g = 10\,m/sec^2$, $\sigma = 0.07\ N/m$, we get

$$k^2 \approx \frac{1000 \times 10}{0.07} = \frac{10^4}{7 \times 10^{-2}} = \frac{10^6}{7}$$

$$k \approx \frac{10^3}{\sqrt{7}}$$

$$\therefore \quad U_{min}^2 = \frac{10^6}{1 \times 10^3} \cdot \frac{10 \times \sqrt{7}}{10^3} + \frac{7 \times 10^{-2} \times 10^3}{1 \times \cancel{10^3}} \times \frac{\cancel{10^3}}{\sqrt{7}}$$

$$\approx 10 \times \sqrt{7} + \sqrt{7} \times 10$$

$$U^2 \approx 20 \times \sqrt{7} \approx 52.91$$

$$U_{min} \approx 7.27\ m/sec$$

$$\lambda_{min} = \frac{2\pi}{k} = \frac{2\pi \times \sqrt{7}}{10^3} = 0.0166\ mm$$

$$\approx 1.66\ cms$$