GOVERNING EQUATIONS

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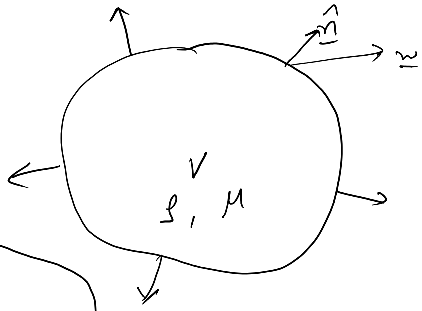
Integral form of Continuity equation:-

If we have any arbitrary quantity f, then



$$\frac{d}{dt}\int_{AR} f\, dV = \int_{AR} \frac{\partial f}{\partial t}\, dV + \int_{AR} n_i w_i f\, dS$$

AR (arbitrary region)

Leibnitz theorem

For continuity equation, $f = \rho$

LHS:
$$\frac{d}{dt}\int \rho\, dV = \frac{d}{dt}\int dm$$

RHS:
$$\int_{AR} \frac{\partial \rho}{\partial t}\, dV + \int_{AR} n_i w_i \rho\, dS$$

But since $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0$, we get

equating LHS with RHS:-

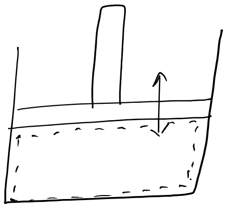
$$\frac{d}{dt}\int_{AR} \rho\, dV = -\int_{AR} \frac{\partial}{\partial x_i}(\rho u_i)\, dV + \int_{AR} n_i w_i \rho\, dS$$

Using divergence theorem, we can simplify this as:

$$\boxed{\frac{d}{dt}\int_{AR} \rho\, dV = -\int n_i \rho(u_i - w_i)\, dS}$$

Rate of change of mass

Integral of mass flow rate through the boundary, & relative to it.



Integral form of the momentum equation:-

Using $f = \rho u_i$, we get

$$\int \frac{\partial(\rho u_i)}{\partial t} dV + \int n_j w_j \rho u_i\, dS$$

Using $f = \rho u_i$,

$$\frac{d}{dt}\int_{AR} \rho u_i \, dV = \int_{AR} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{AR} n_j w_j \rho u_i \, dS$$

from N.S. equations;

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j) + \rho F_i + \frac{\partial}{\partial x_j}(T_{ij})$$

F_i : Body force per unit mass

T_{ij} : Stress tensor

$$= -p\,\delta_{ij} + \tau_{ij}$$

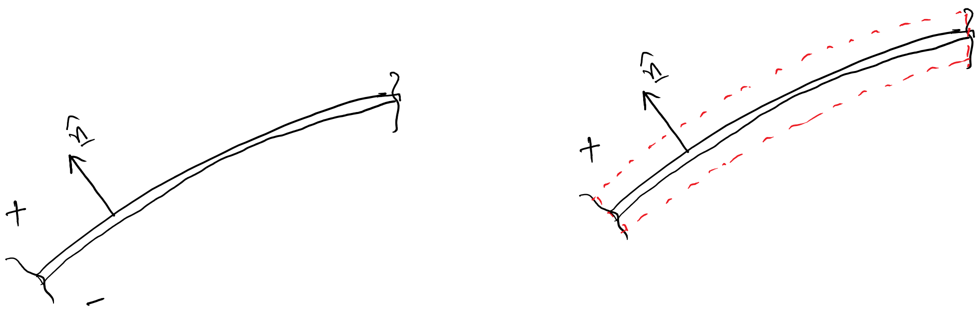
$$= -p\,\delta_{ij} + 2\mu S_{ij}$$

$$= -p\,\delta_{ij} + \mu(\nabla u + \nabla u^T)_{ij}$$

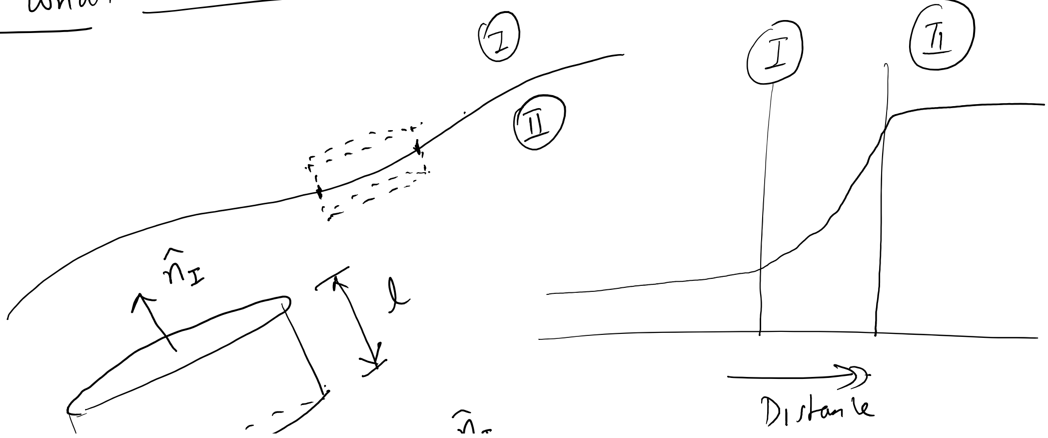
$$\Rightarrow \frac{d}{dt}\int_{AR} \rho u_i \, dV = \int_{AR}\left\{ -\frac{\partial}{\partial x_j}(\rho u_i u_j) + \rho F_i + \frac{\partial}{\partial x_j}(T_{ij})\right\} dV + \int_{AR} n_j w_j \rho u_i \, dS$$

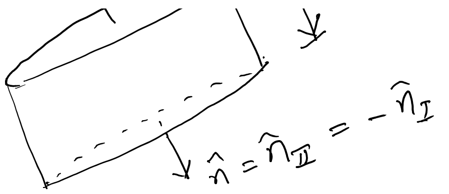
$$= \int_{AR} \rho F_i \, dV - \int_{AR} n_j \rho u_i u_j \, dS + \int n_j T_{ij}\, dS + \int n_j w_j \rho u_i \, dS$$

$$\boxed{\frac{d}{dt}\int_{AR} \rho u_i \, dV = \int_{AR} \rho F_i \, dV - \int_{AR} n_j \rho (u_j - w_j) u_i \, dS + \int_{AR} n_j T_{ij} \, dS}$$



Jump conditions at interfaces:-





$$\frac{d}{dt}\int \rho \, dV = -\int_{II} \hat{n}\cdot(\underline{u}-\underline{w})\rho \, dS + \int \hat{n}\cdot(\underline{u}-\underline{w})\rho \, dS$$
$$+ \int \text{Side surface of cylinder}$$

To arrive at an interface jump condition, we shrink the cylinder to zero height, i.e., $l \to 0$:

$\Rightarrow$ Volume integrals disappear & $\int \text{Side Surface} \to 0$

$$\Rightarrow 0 = -\int_{II} \hat{n}\cdot(\underline{u}-\underline{w})\rho \, dS + \int_{I} \hat{n}\cdot(\underline{u}-\underline{w})\rho \, dS$$

$$\Rightarrow \boxed{\left[\hat{n}\cdot(\underline{u}-\underline{w})\rho\right]_{II} = \left[\hat{n}\cdot(\underline{u}-\underline{w})\rho\right]_{I}}$$

$n\cdot(\underline{u}-\underline{w})$: Normal component of relative velocity at the interface.

Stress conditions:- (without surface tension) :-

$$\frac{d}{dt}\int_{AR} \rho u_i \, dV = \int_{AR} \rho f_i \, dV - \int_{AR} n_j \rho (u_j - w_j) u_i \, dS + \int_{AR} n_j T_{ij} \, dS$$

Assuming that there are no sources of momentum within the interface, shrinking the interface to zero volume gives:-

$$0 = 0 - \int_{II} n_j \rho (u_j - w_j) u_i \, dS + \int_{I} n_j \rho (u_j - w_j) u_i \, dS$$

$$\int n_j T_{ij} \, dS - \int n_j T_{ij} \, dS$$

$\int_{II}$

$$+ \int_{II} n_j \underline{T_{ij}} \, ds - \int_{I} n_j T_{ij} \, ds$$

$$\Rightarrow 0 = \int_{II} \left\{ \rho \underline{u} \, (u_n - w_n) - \underline{\hat{n}} \cdot \underline{\underline{T}} \right\} ds$$

$$- \int_{I} \left\{ \rho \underline{u} \, (u_n - w_n) - \underline{\hat{n}} \cdot \underline{\underline{T}} \right\} ds$$

$$\Rightarrow \left\{ \rho \underline{u} \, (u_n - w_n) - \underline{\hat{n}} \cdot \underline{\underline{T}} \right\}_{II} = \left\{ \rho \underline{u} \, (u_n - w_n) - \underline{\hat{n}} \cdot \underline{\underline{T}} \right\}_{I}$$

Tangential Component:-

across the interface

① No-slip condition

$$\Rightarrow \quad \underline{u} \cdot \underline{\hat{t}} \Big|_{II} = \underline{u} \cdot \underline{\hat{t}} \Big|_{I}$$

$$\Rightarrow \quad u_{t_{I}} = u_{t_{II}}$$

② $$\left(\underline{\hat{n}} \cdot \underline{\underline{T}} \cdot \underline{\hat{t}} \right)_{I} = \left(\underline{\hat{n}} \cdot \underline{\underline{T}} \cdot \underline{\hat{t}} \right)_{II}$$

If $\underline{\underline{T}} = -p \underline{\underline{I}} + \underline{\underline{\tau}}$

$$n_j T_{ij} = n_j \left(-p \delta_{ij} + \tau_{ij} \right)$$

$$= -p n_i + n_j \tau_{ij}$$

or $\underline{n} \cdot \underline{\underline{T}} = -p \underline{\hat{n}} + \underline{\hat{n}} \cdot \underline{\underline{\tau}}$

We get

$$\left(\underline{\hat{n}} \cdot \underline{\underline{\tau}} \cdot \underline{\hat{t}} \right)_{I} = \left(\underline{\hat{n}} \cdot \underline{\underline{\tau}} \cdot \underline{\hat{t}} \right)_{II}$$

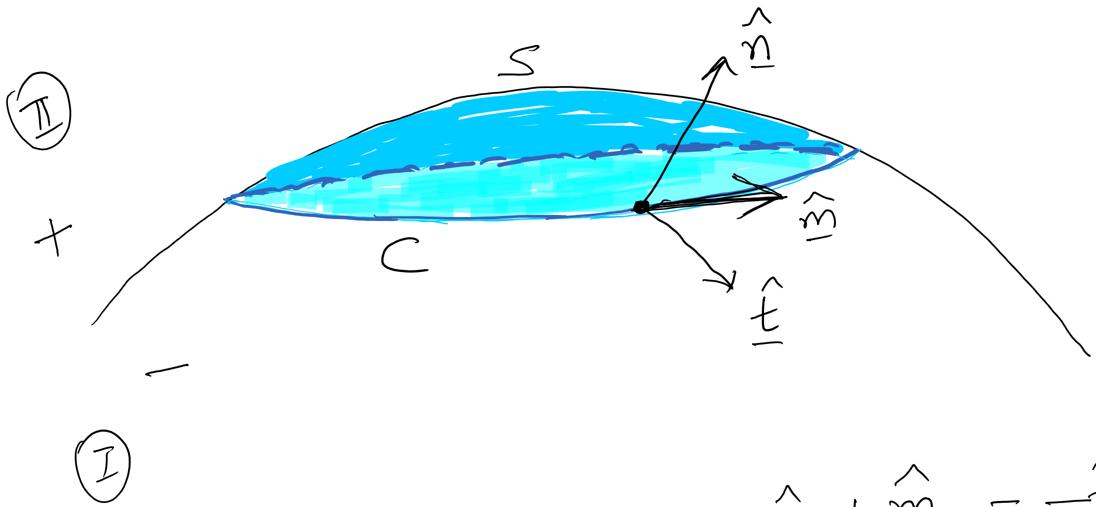
The shear stress is continuous across an interface.

Normal Component :-

$$\left\{(u_n - w_n)\rho u_n + p - \tau_{nn}\right\}_{I} = \left\{(u_n - w_n)\rho u_n + p - \tau_{nn}\right\}_{II}$$

$$\tau_{nn} = \underline{\hat{n}} \cdot \underline{\underline{\tau}} \cdot \underline{\hat{n}}$$

Jump conditions at an interface with surface tension :-



$$\underline{\hat{n}} \times \underline{\hat{m}} = -\underline{\hat{t}}$$

$$\frac{d}{dt}\int_V \rho u_i \, dV = \int_V \rho f_i \, dV - \int_S n_j \rho (u_j - w_j) u_i \, dS + \int_S n_j T_{ij} \, dS + \int_C \sigma \hat{t} \, dl$$

l indicates arc-length & so dl is a length increment along the curve C.

$\sigma \rightarrow$ surface tension.

$\int \rho f_i \, dV$: body force acting within V.

$\int_S n_j T_{ij}\, dS$: Hydrodynamic force

$\int_C \sigma \hat{t}\, dl$: Surface tension force exerted on perimeter

Like before, we take the height of our control volume to shrink to zero, we get

$$0 = 0 - \int_{+} \rho \hat{n}\cdot(\underline{u}-\underline{w})\underline{u}\, dS + \int_{-} \rho \hat{n}\cdot(\underline{u}-\underline{w})\underline{u}\, dS$$

$$+ \int_{+} (\hat{\underline{n}}\cdot \underline{\underline{T}})\, dS - \int (\hat{\underline{n}}\cdot\underline{\underline{T}})\, dS$$

$$+ \oint_C \sigma \hat{t}\, dl$$

$\int_{+} (\hat{n}\cdot\underline{\underline{T}})\, dS$: Hydrodynamic ~~stress~~ force exerted on the interface by the upper $+$ fluid.

$\int_{-} (\hat{n}\cdot\underline{\underline{T}})\, dS$: Hydrodynamic force exerted by the lower fluid $-$ on the interface.

$$\int_C \sigma \hat{t}\, dl = \int_S \{\nabla_s\sigma - \sigma\hat{n}(\nabla_s\cdot\hat{n})\}\, dS$$

∇_s : Surface gradient operator $= \frac{\partial}{\partial s}$ → arc-length derivative

$$\nabla_s = [\underline{\underline{I}} - \hat{\underline{n}}\hat{\underline{n}}]\cdot\nabla = \nabla - \hat{\underline{n}}\frac{\partial}{\partial \hat{\underline{n}}}$$

Cartesian gradient operator

$$\Rightarrow \int_{\oplus} \rho \, \underline{\hat{n}} \, (\underline{u} - \underline{w}) \, \underline{u} \, dS - \int_{\ominus} \rho \, \underline{\hat{n}} \, (\underline{u} - \underline{w}) \, \underline{u} \, dS$$

$$= \int_{\oplus} (\underline{\hat{n}} \cdot \underline{\underline{T}}) \, dS - \int_{\ominus} (\underline{\hat{n}} \cdot \underline{\underline{T}}) \, dS$$

$$+ \int_{S} \left\{ \nabla_s \sigma - \sigma \, \hat{n} \, (\nabla_s \cdot \underline{\hat{n}}) \right\} dS$$

Since surface is arbitrary:-

$$\left[\left(\rho \, \hat{n} \, (\underline{u} - \underline{w}) \, \underline{u} \right) - \left(\underline{\hat{n}} \cdot \underline{\underline{T}} \right) \right]_{\oplus} - \left[\left(\rho \, \hat{n} \, (\underline{u} - \underline{w}) \underline{u} \right) - \left(\underline{\hat{n}} \cdot \underline{\underline{T}} \right) \right]_{\ominus}$$

$$= \left\{ \nabla_s \sigma - \sigma \, \hat{n} \, (\nabla_s \cdot \underline{\hat{n}}) \right\}$$

This condition is valid at every point on the surface.

Tangential conditions:-

(1) No-slip at the interface.

$$\Rightarrow \quad u_t \Big|_{+} = u_t \Big|_{\ominus}$$

Tangential velocity is continuous

(2) $$\left(-\underline{\hat{n}} \cdot \underline{\underline{T}} \cdot \underline{\hat{t}} \right)_{\oplus} - \left(-\underline{\hat{n}} \cdot \underline{\underline{T}} \cdot \underline{\hat{t}} \right)_{\ominus} = \underline{\hat{t}} \cdot \nabla_s \sigma$$

Using $\underline{\underline{T}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$, we get

$$\boxed{(\hat{n} \cdot \tau \cdot \underline{\hat{t}}) - (\underline{\hat{n}} \cdot \underline{\underline{\tau}} \cdot \underline{\hat{t}})_{\ominus} = -\underline{\hat{t}} \cdot \nabla_s \sigma}$$

$$\boxed{(\hat{n}\cdot\underline{\underline{\tau}}\cdot\hat{t})_{\oplus} - (\hat{n}\cdot\underline{\underline{\tau}}\cdot\hat{t})_{\ominus} = -\underline{t}\cdot\nabla_s\sigma}$$

Marangoni Stress

Gradients in surface tension drives a flow. This is called Marangoni flow or Marangoni convection.

Normal stress balance:-

$$\{\rho u_n(u_n - w_n)\}_+ - (\hat{n}\cdot\underline{\underline{T}}\cdot\hat{n})_+ - \{\rho u_n(u_n-w_n)\}_- + (\hat{n}\cdot\underline{\underline{T}}\cdot\hat{n})_- = -\sigma(\nabla_s\cdot\hat{n})$$

for a stationary system (no flow), we get

$$(\hat{n}\cdot\underline{\underline{T}}\cdot\hat{n})_{\ominus} - (\hat{n}\cdot\underline{\underline{T}}\cdot\hat{n})_+ = -\sigma(\nabla_s\cdot\hat{n})$$

where $\hat{n}\cdot\underline{\underline{T}} = -p\hat{n}$

$$\Rightarrow \quad p_{\ominus} - p_{\oplus} = -\sigma(\nabla_s\cdot\hat{n})$$

$$\Rightarrow \quad \boxed{p_{below} - p_{above} = -\sigma(\nabla_s\cdot\hat{n})}$$

$-(\nabla_s\cdot\hat{n}) = \mathcal{H}$: Mean curvature

$= \frac{1}{R_1} + \frac{1}{R_2}$