Relationship between $\oint_C \sigma \hat{t} \, dl$ and

<u>a surface integral:-</u>

We start with the Stokes theorem:-

$$\oint_C \underline{F} \cdot \underline{dl} = \int_S \hat{n} \cdot (\nabla \times \underline{F}) \, dS$$

Along a contour C, we have $\underline{dl} = \underline{\hat{m}} \, dl$

$$\Rightarrow \oint_C \underline{F} \cdot \underline{\hat{m}} \, dl = \int_S \underline{\hat{n}} \cdot (\nabla \times \underline{F}) \, dS$$

Now we define $\underline{F} = \underline{f} \times \underline{b}$ where $\underline{b}$ is an arbitrary constant vector.

$$\Rightarrow \oint_C (\underline{f} \times \underline{b}) \cdot \underline{\hat{m}} \, dl = \int_S \underline{\hat{n}} \cdot (\nabla \times (\underline{f} \times \underline{b})) \, dS$$

Using the vector identities:-

$$(\underline{f} \times \underline{b}) \cdot \underline{m} = -\underline{b} \cdot (\underline{f} \times \underline{m}) \text{ , and}$$

$$\nabla \times (\underline{f} \times \underline{b}) = \underline{f}(\nabla \cdot \underline{b}) - \underline{b}(\nabla \cdot \underline{f}) + \underline{b} \cdot \nabla \underline{f} - \underline{f} \cdot \nabla \underline{b}$$

$$= -\underline{b}(\nabla \cdot \underline{f}) + \underline{b} \cdot \nabla f$$

since $\underline{b}$ is a constant vector.

We get

$$\int \hat{n} \cdot \{ -\underline{b}(\nabla \cdot f) + \underline{b} \cdot \nabla f \} dS$$

We get

$$-\oint_C \underline{b}\cdot(\underline{f}\times\underline{m})\, dl = \int_S \hat{\underline{n}}\cdot\{-\underline{b}(\nabla\cdot\underline{f}) + \underline{b}\cdot\nabla\underline{f}\}\, dS$$

$$\Rightarrow \underline{b}\cdot\oint_C(\underline{f}\times\underline{m})\, dl = \underline{b}\cdot\int_S\{\hat{\underline{n}}(\nabla\cdot\underline{f}) - \nabla\underline{f}\cdot\hat{\underline{n}}\}\, dS$$

Since $\underline{b}$ is arbitrary, we can write

$$\oint_C(\underline{f}\times\underline{m})\, dl = \int_S [\hat{n}(\nabla\cdot\underline{f}) - \nabla\underline{f}\cdot\hat{\underline{n}}]\, dS$$

Now we let $\underline{f} = \sigma\hat{\underline{n}}$, and since

$$\underline{n}\times\underline{m} = -\hat{t}$$

$$\oint_C \sigma(\underline{n}\times\underline{m})\, dl = \int_S\{\hat{\underline{n}}[\nabla\cdot(\sigma\hat{\underline{n}})] - \nabla(\sigma\hat{\underline{n}})\cdot\hat{\underline{n}}\}\, dS$$

$$\Rightarrow -\oint_C \sigma\hat{t}\, dl = \int_S\Big[\hat{\underline{n}}\{\hat{\underline{n}}\cdot\nabla\sigma + \sigma(\nabla\cdot\hat{\underline{n}})\} - \nabla\sigma(\hat{\underline{n}}\cdot\hat{\underline{n}}) - \sigma(\nabla\hat{\underline{n}}\cdot\hat{\underline{n}})\Big]\, dS$$

$$\nabla\sigma(\hat{\underline{n}}\cdot\hat{\underline{n}}) = \nabla\sigma$$

$$\nabla\hat{n}\cdot\hat{n} = \frac{1}{2}\nabla(\hat{\underline{n}}\cdot\hat{\underline{n}}) = \frac{1}{2}\nabla(1) = 0$$

$\hat{\underline{n}}\cdot\nabla\sigma = 0$ Since $\nabla\sigma$ will be along the interface

We get

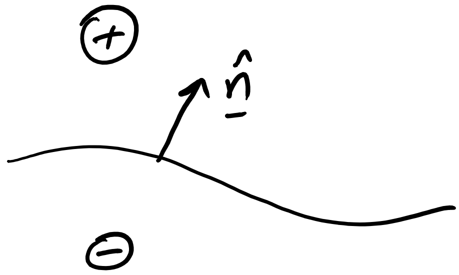
$$-\oint_C \sigma\hat{t}\, dl = \int_S[\sigma\hat{n}(\nabla\cdot\hat{n}) - \nabla\sigma]\, dS$$

$$\oint \hat{t} \ldots = \int[\nabla\sigma \quad \sigma\hat{n}(\nabla\cdot\hat{n})]\, dS$$

$$\text{or} \quad \oint_C \sigma \hat{t}\, dl = \iint_S \left[\nabla\sigma - \sigma\hat{n}(\nabla\cdot\hat{\underline{n}})\right] ds$$

FLUID STATICS:-

In a static scenario,



$$\underline{\underline{T}} = -p\,\underline{\underline{I}}$$

& $\hat{\underline{n}}\cdot\underline{\underline{T}}\cdot\hat{\underline{n}} = -p$

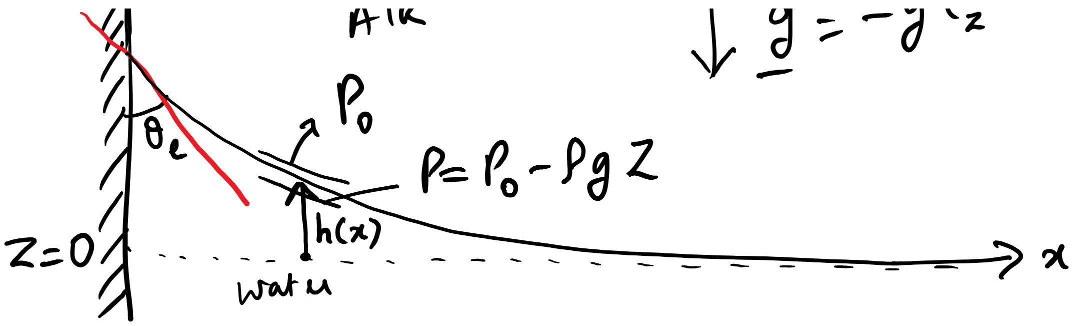
Normal stress balance equation reduces to

$$p_{\ominus} - p_{\oplus} = -\sigma(\nabla\cdot\hat{\underline{n}})$$

For fluid statics, σ needs to be constant, else, if $\nabla\sigma \neq 0$, then there will be viscous stresses needed to balance the tangential stress balance $\Rightarrow$ flow will be generated for variable surface tension cases.

Meniscus at a vertical plate:-

$\downarrow \underline{g} = -g\,\hat{e}_z$



Given: θ_e, σ, ρ_w, ρ_a, g
$\sigma \rightarrow$ constant

Assume: $\rho_a << \rho_w \Rightarrow \rho = \rho_w - \rho_a \approx \rho_w$

We know that

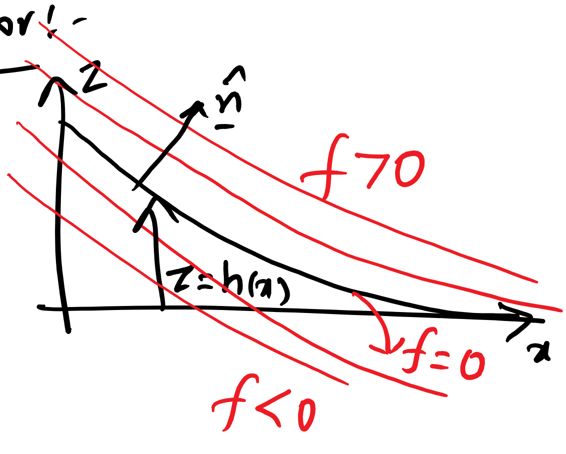
$$p_{\ominus} - p_{\oplus} = -\sigma(\nabla \cdot \hat{\underline{n}})$$

Here $p_{\oplus} = p_0$: constant atmosphere pressure

$$p_{\ominus} = p_0 - \rho g z = p_0 - \rho g h$$

Define the interface as $z = h(x)$.

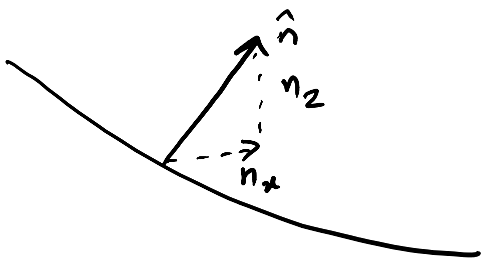
To define normal vector:



Define $f(x,z) = z - h(x)$

The interface is given by

$$f(x,z) = 0$$



$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial z}\hat{k}$$

$$\frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial z}$$

$$= -\frac{dh}{dx}\,\hat{i} + \hat{k}$$

$$\therefore \boxed{\hat{n} = \frac{\hat{k} - \left(\frac{dh}{dx}\right)\hat{i}}{\sqrt{1+\left(\frac{dh}{dx}\right)^2}}} \qquad h' = \frac{dh}{dx}$$

$$\nabla\cdot\hat{n} = \frac{\partial}{\partial x}(n_x) + \frac{\partial}{\partial z}(n_z) \qquad \text{where}$$

$$n_x = \frac{-h'}{\sqrt{1+h'^2}} \; ; \quad n_z = \frac{1}{\sqrt{1+h'^2}}$$

$$\therefore \nabla\cdot\underline{\hat{n}} = -\left\{\frac{\sqrt{1+h'^2}\,h'' - h'\cdot\frac{1}{\cancel{2}\sqrt{1+h'^2}}\cdot\cancel{2}h'h''}{(1+h'^2)}\right\} + 0$$

$$= -\frac{\left[h''(1+h'^2) - h'^2h''\right]}{(1+h'^2)^{3/2}}$$

$$\boxed{\nabla\cdot\underline{\hat{n}} = \frac{-h''}{(1+h'^2)^{3/2}}}$$

$$(P_0 - \rho g h) - P_0 = -\sigma\frac{h''}{(1+h'^2)^{3/2}}$$

$$\Rightarrow \boxed{\rho g h = \frac{\sigma h''}{(1+h'^2)^{3/2}}}$$

Second order nonlinear differential equation for $h(x)$.

Boundary conditions:-

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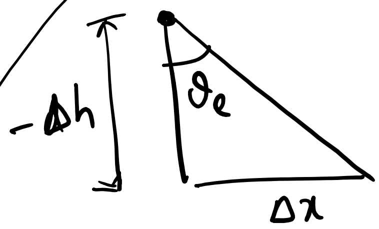
As $x \to \infty$, $h = 0$

At $x = 0$; $\frac{dh}{dx}(x=0) = -\cot(\theta_e)$



$$\tan\theta_e = \frac{\Delta x}{(-\Delta h)}$$

(or) $\frac{dh}{dx} = -\cot\theta_e$

For small slopes,

$$\frac{dh}{dx} << 1$$

$$\Rightarrow 1 + \left(\frac{dh}{dx}\right)^2 \approx 1$$

Equation becomes:

$$\sigma h'' = \rho g h$$

(or) $h'' = \frac{\rho g}{\sigma} \cdot h$

Note: $\frac{\rho g}{\sigma} = \frac{1}{l_c^2}$

$$\Rightarrow \boxed{h'' = \frac{h}{l_c^2}}$$

Integrating twice, we get

$$\boxed{h(x) = l_c \cdot \cot(\theta_e)\, e^{-x/l_c}}$$