Capillary rise through tubes:-

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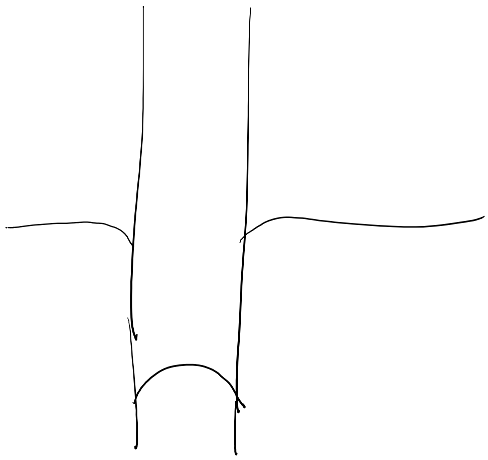

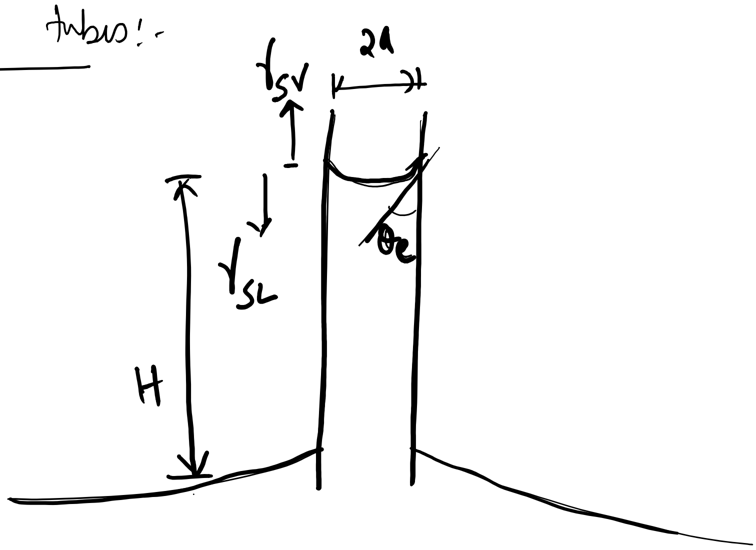

Spreading parameter: $S = \gamma_{SV} - (\gamma_{SL} + \gamma_{LV})$

We can also define a new parameter:

$$\underline{I = \gamma_{SV} - \gamma_{SL}} = \gamma_{LV} \cos\theta_e$$

σ

$I > 0$, if $\theta_e < 90°$, else $I < 0$

We said, $S > 0 \Rightarrow$ Complete spreading.

I : Impregnation parameter.

It is much easier to obtain $I > 0$ than $S > 0$. $\Rightarrow$ In nature we usually encounter partial spreading.

$\rightarrow$ Water column

Total energy of the system:-

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Total energy of the system:-

$$E = (\gamma_{SL} - \gamma_{SV})\, 2\pi a \cdot H + \rho g \pi a^2 \frac{H^2}{2}$$

$$= \underbrace{-I \cdot 2\pi a H}_{\text{surface energy}} + \underbrace{\rho g \pi a^2 \frac{H^2}{2}}_{\text{gravitation P.E}}$$

Height chosen can be obtained by minimizing energy:-

$$\frac{\partial F}{\partial H} = 0 \Rightarrow -2\pi a \cdot I + \rho g \pi a^2 H = 0$$

$$\Rightarrow H = \frac{2\cancel{\pi a}\, I}{\rho g \cancel{\pi}\, a^{\cancel{2}}} = \frac{2I}{\rho g a}$$

But $I = \sigma \cos\theta_e$

$$\Rightarrow \boxed{H = \frac{2\sigma\cos\theta_e}{\rho g a}}$$: Jurin's Law

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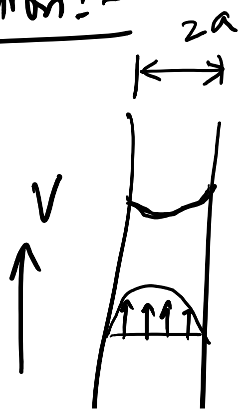
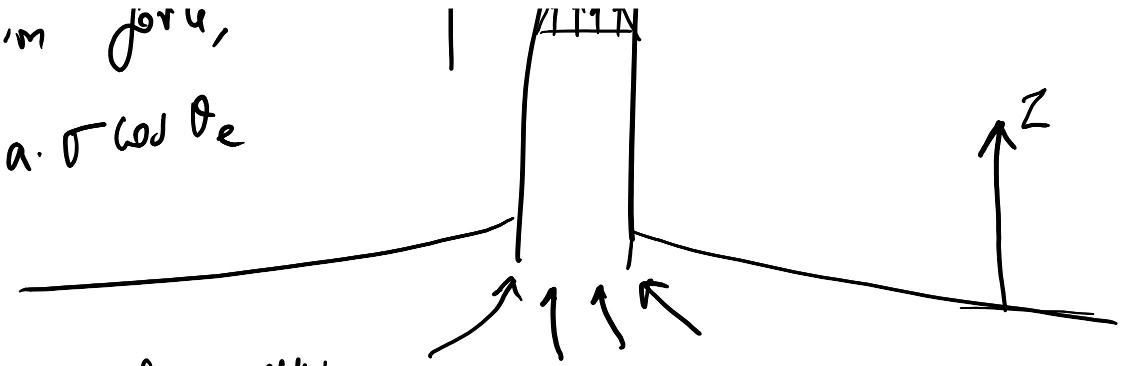
Dynamics of Impregnation:-

Force causing the fluid to rise is surface tension force,



Surface tension force,

$F = 2\pi a \cdot \sigma \cos\theta_e$



Forces opposing this rise! - • Gravity

• Viscosity

• Inertia

$$\frac{d}{dt}(MV) = F - F_\mu - \underbrace{W}_{\text{weight } = Mg}$$

$\underline{F_\mu}$: Assuming the flow is fully developed, (this happens only after a viscous time, $t_\nu = \frac{a^2}{\nu}$),

$$u(r) = 2\dot{z}\left(1 - \frac{r^2}{a^2}\right)$$

$\dot{z} = V$: average velocity

Shear stress : $\tau = \mu \left.\frac{\partial u}{\partial r}\right|_{r=a} = -\frac{4\mu}{a}\dot{z}$

$= -\frac{4\mu}{a} \cdot V$

$$f_\mu = 2\pi \cancel{a} \cdot z \cdot -\frac{4\mu}{\cancel{a}} \cdot V$$

$$= -8\pi \mu V z$$

$$\frac{d}{dt}(Mv) = \underbrace{2\pi a \cdot \gamma \cos\theta}_{F} - \underbrace{8\pi\mu V z}_{F_\mu} - \underbrace{Mg}_{W}$$

Viscous regime:- valid for times $> t_v$

Ignoring inertia since the fluid rises slowly at later times, we get

$$F - F_\mu - W = 0$$

$$\Rightarrow F - W = F_\mu$$

For thin capillary tubes, $F \propto$ Surface area
$W \propto$ Volume

$$\Rightarrow F \gg W$$

$$\Rightarrow F \simeq F_\mu$$

$$\Rightarrow 2\pi a \cdot \sigma \cos\theta_e = 8\pi\mu V z$$

$$= 8\pi\mu\, z \cdot \dot{z}$$

$$= 8\pi\mu \frac{d}{dt}\left(\frac{z^2}{2}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{z^2}{2}\right) = \frac{2\cancel{\pi} a\, \sigma \cos\theta_e}{8\cancel{\pi}\mu}$$

$$\frac{d}{dt}(z^2) = \frac{1}{2}\frac{a\sigma\cos\theta_e}{\mu}$$

Integration:- $$z^2 \simeq \frac{1}{2}\frac{a\sigma\cos\theta_e}{\mu}\cdot t$$

or $$\boxed{z \sim \left(\frac{1}{2}\frac{\sigma a\cos\theta_e}{\mu}\right)^{1/2}\cdot\sqrt{t}}$$

Washburn Law/formula

Inertial regime:-

This is valid at short times, or;

$$t << t_v$$

Also at short times, W can be ignored.

$$\frac{d}{dt}(Mv) = F \quad \text{(Ignoring viscous forces)}$$

$$\frac{d}{dt}\left(\rho\cdot\cancel{\pi}a^{\cancel{2}}z(t)\cdot\dot{z}\right) = 2\cancel{\pi}\cancel{a}\cdot\sigma\cos\theta_e$$

$$\frac{d}{dt}(z\cdot\dot{z}) = \frac{2\sigma\cos\theta_e}{\rho a}$$

or $$\frac{d}{dt}(z\underbrace{\dot{z}}_{\frac{dz}{dt}}) \sim \frac{2\sigma\cos\theta_e}{\rho a}$$

$$\frac{dz}{dt}$$

$$\frac{d^2}{dt^2}\left(\frac{z^2}{2}\right) \sim \frac{2\sigma\cos\theta_e}{\rho a}$$

$$z \cdot \dot{z} = \frac{2\sigma\cos\theta_e}{\rho a} \cdot t$$

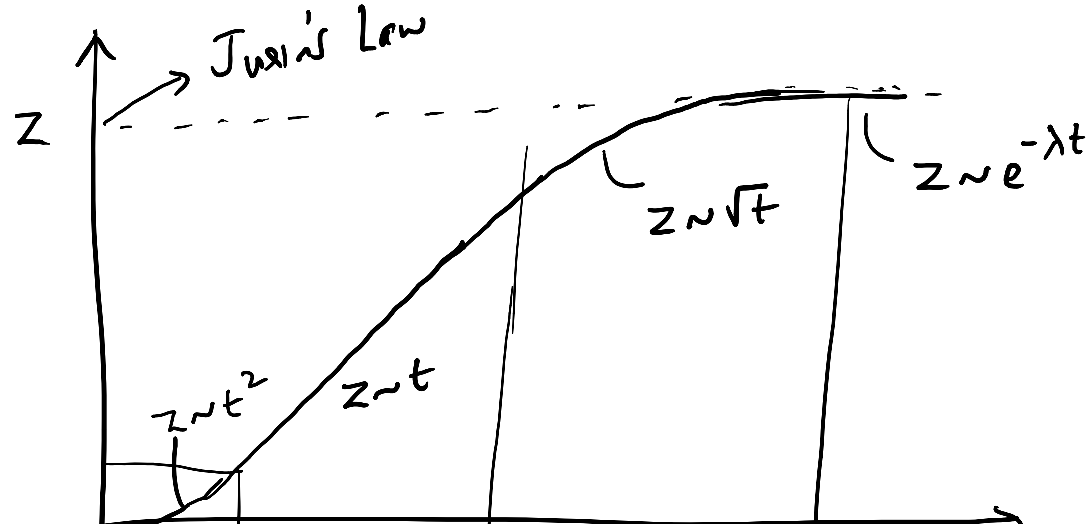
$$\frac{d}{dt}\left(\frac{z^2}{2}\right) = \frac{2\sigma\cos\theta_e}{\rho a} \cdot t$$

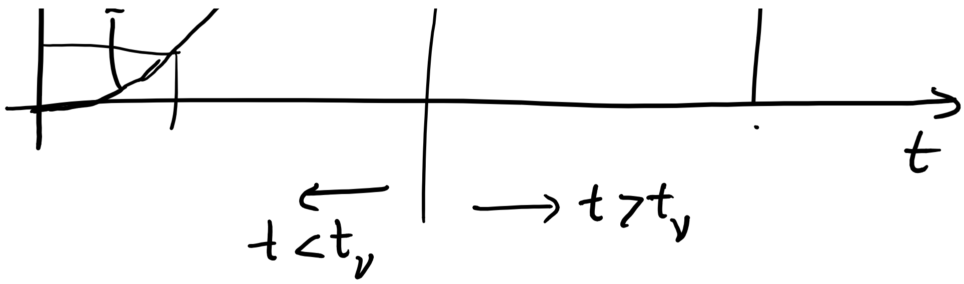
$$d\left(\frac{z^2}{2}\right) = \frac{2\sigma\cos\theta_e}{\rho a} \cdot t\,dt$$

$$\frac{z^2}{2} = \frac{\cancel{2}\sigma\cos\theta_e}{\rho a} \cdot \frac{t^2}{\cancel{2}}$$

$$\Rightarrow \quad z^2 = \frac{2\sigma\cos\theta_e}{\rho a} \cdot t^2$$

$$\Rightarrow \quad \boxed{z(t) = \left(\frac{2\sigma\cos\theta_e}{\rho a}\right)^{1/2} \cdot t}$$





$$\delta \sim \sqrt{\nu t} \sim \sqrt{\nu t_\nu} \sim \sqrt{\nu \frac{a^2}{\nu}} = a$$