# Rayleigh-Plateau Instability:-

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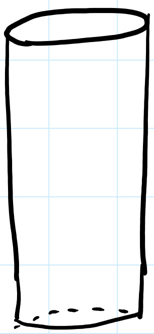
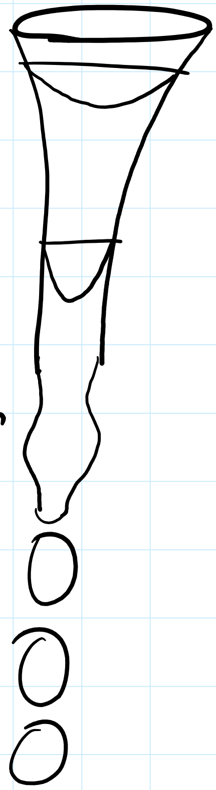
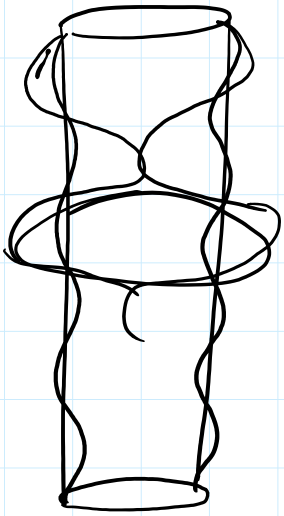




(Q): If there is a thin column of fluid suspended in air, will it break into drops?

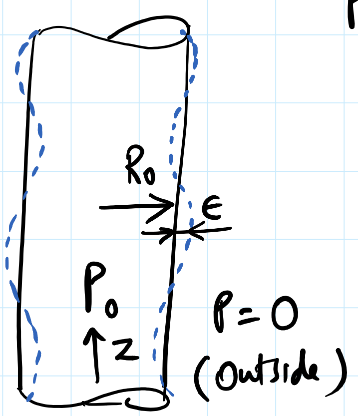


$$h(x,t) = \sum (\ ) e^{i\omega t} \cdot e^{ikx} + c.c$$

$$= \mathrm{Re}\left\{ \sum (\ ) e^{i\omega t} e^{ikx} \right\}$$

$$\underbrace{\sin(kx)}_{\text{wave number}} \rightarrow \underbrace{\sin\left(\frac{x}{\lambda}\right)}_{\text{wave length}}$$

## Undisturbed column:-




$$P_0 - 0 = \sigma(\nabla\cdot\hat{n})$$

$$= \frac{\sigma}{R_0}$$

... the column:-

Introduce a small disturbance to the Column:-

$$R = R_0 + \epsilon e^{\omega t + ikz}, \quad \text{wt} \quad \epsilon << R_0$$

**Goal:** Determine relation between $\omega$ & $K$!

We consider scenario's where the flow is

- axisymmetric $\left(\frac{\partial(\cdot)}{\partial\theta} = 0\right)$
- No flow in $\theta$- direction $(u_\theta = 0)$

**Governing equations:-**

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \cancel{\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}} + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}} + u_z\frac{\partial u_r}{\partial z} - \cancel{\frac{u_\theta^2}{r}} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu$$

$$\cancel{\frac{\partial u_\theta}{\partial t}} + u_r\cancel{\frac{\partial u_\theta}{\partial r}} + \cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}} + \cancel{u_z \cdot \frac{\partial u_\theta}{\partial z}} + \cancel{\frac{u_r u_\theta}{r}} = \cancel{-\frac{1}{r}\cdot\frac{1}{\rho}\frac{\partial P}{\partial\theta}} \cancel{+ \nu(}$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}} + u_z\cdot\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z}$$

**Reduced governing equations!-**

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial u_r}{\partial t} + \underline{u_r\frac{\partial u_r}{\partial r}} + \underline{u_z\frac{\partial u_r}{\partial z}} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \cancel{\nu(\quad)} \rightarrow 0$$

$$\frac{\partial u_z}{\partial t} + \underline{u_r\frac{\partial u_z}{\partial r}} + \underline{u_z\frac{\partial u_z}{\partial z}} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \cancel{\nu(\quad)} \rightarrow 0$$

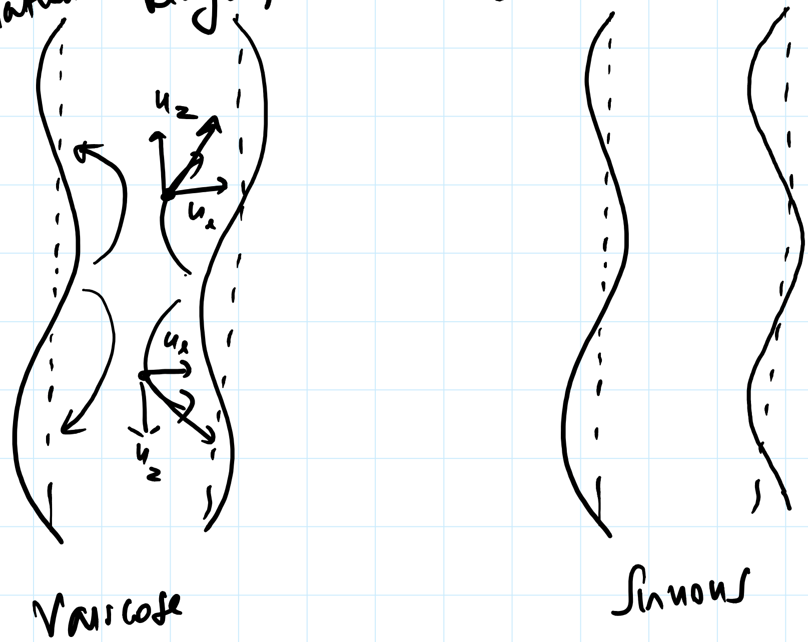
$$\frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} + u_z \qquad \rho\, \partial z$$

non-linear terms

**Assumption** : • Viscous terms are negligible

• This was what was done by Rayleigh.

This is more relevant for plateau - Rayleigh instability




Since,

$$R = R_0 + \epsilon\, e^{\omega t + ikz}$$

$$u_x(x,z,t) = V(x)\, e^{\omega t + ikz}$$

$$u_z(x,z,t) = W(x)\, e^{\omega t + ikz}$$

$$p(x,z,t) = P(x)\, e^{\omega t + ikz}$$

Because $\epsilon << R_0$, the perturbations $V(x)$, $W(x)$ & $P(x)$ are small, ie; products of these perturbations can be ignored. So we can retain only linear terms in the governing ...

equation. Substituting, we get :-

$$\frac{\partial u_r}{\partial t} \longrightarrow V(r)\,\omega \cdot e^{\omega t + ikz}$$

$$\frac{\partial p}{\partial r} \longrightarrow \frac{dP}{dr}\, e^{\omega t + ikz}$$

$$\frac{\partial p}{\partial z} \longrightarrow P(r)\; ik\, e^{\omega t + ikz}$$

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$$\omega\, V(r)\, \cancel{e^{\omega t + ikz}} = \frac{-1}{\rho}\frac{dP}{dr}\, \cancel{e^{\omega t + ikz}}$$

$$\omega\, W(r)\, \cancel{e^{\omega t + ikz}} = \frac{-1}{\rho}\, ik\, P\, \cancel{e^{\omega t + ikz}}$$

$$\frac{dV}{dr} + \frac{V}{r} + ikW = 0$$

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Simplified equations :-

$$\omega V = \frac{-1}{\rho}\frac{dP}{dr}$$

$$\omega W = -\frac{ik}{\rho} P$$

$$\frac{dV}{dr} + \frac{V}{r} + ikW = 0$$

Eliminating $W(r)$ & $P(r)$, one single equation for $V(r)$ can be obtained as follows :-

$$\boxed{r^2 \frac{d^2 V}{dr^2} + r\frac{dV}{dr} - \left(1 + (rk)^2\right) V = 0}$$

$x \frac{\;}{dr^2} + \quad \frac{\;}{dr}$

An equation of the form:

$$r^2 \frac{d^2 V}{dr^2} + r \frac{dV}{dr} - (m^2 + k^2 r^2)\, V = 0$$

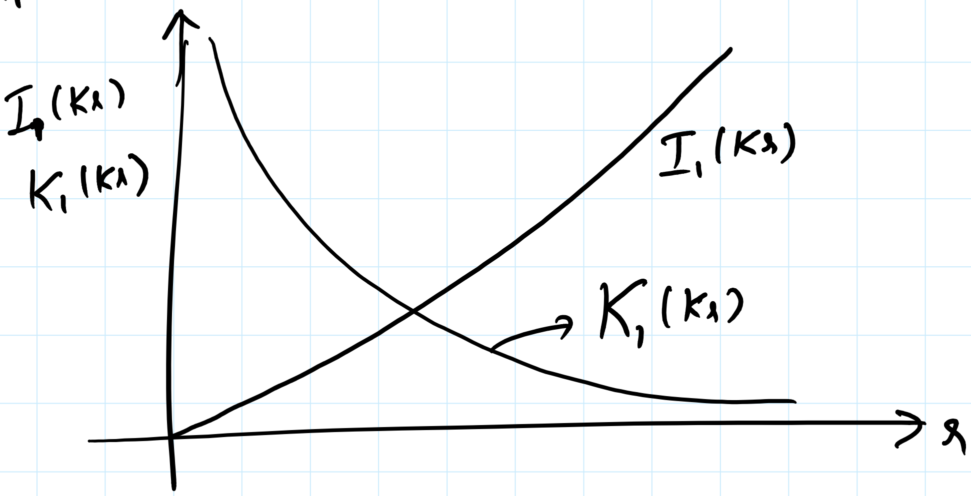
has a general solution:-

$$V(r) = C_1 I_m(kr) + C_2 K_m(kr)$$

Modified Bessel functions of first & second kind.

$$\therefore V(r) = C_1 I_1(kr) + C_2 K_1(kr)$$

But $K_1(kr) \to \infty$ as $r \to 0$, we require $C_2 = 0$.




$$V(r) = C_1 I_1(kr)$$

$$\frac{dP}{dr} = -\rho\omega \cdot C_1 \cdot I_1(kr)$$

$$P(r) = -\rho\omega C_1 \cdot \int I_1(kr)\, dr$$

Using the result $\frac{d}{dx}\left(I_0(x)\right) = I_1(x)$, we get

$$x \to kr$$

$$\frac{d}{dx} = \frac{d}{dr}\cdot\frac{1}{k}$$

$$P(r) = -\frac{\rho\omega C_1}{k}\cdot I_0(kr)$$

Similarly, $W(r) = \frac{-ik}{\rho\omega}\cdot P$

$$= \frac{-i\cancel{k}}{\cancel{\rho\omega}}\cdot\times\ \frac{-\cancel{\rho\omega}\, C_1}{\cancel{k}}\cdot I_0(kr)$$

$$W(r) = i C_1 I_0(kr)$$

<u>Summary:</u>

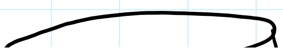
$$U(r) = C_1 I_1(kr)$$
$$W(r) = i C_1 I_0(kr)$$
$$P(r) = -\frac{\rho\omega C_1}{k} I_0(kr)$$

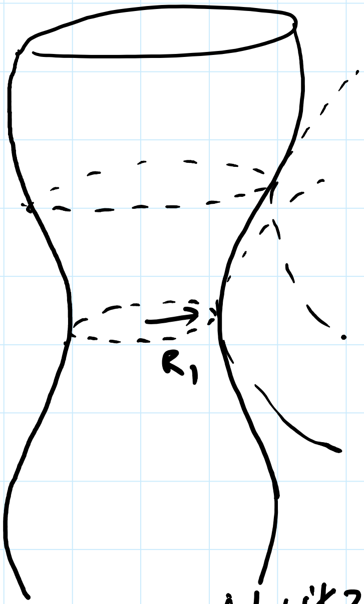

$$\frac{\partial R}{\partial t} = u_r\Big|_{\text{at } r=R}$$

$$\epsilon\omega\, \cancel{e^{\omega t+ikz}} = U(r=R)\, \cancel{e^{\omega t+ikz}}$$

$$\boxed{\epsilon\omega = C_1 I_1(kR)} \quad \text{——— (i)}$$

$$\Delta P = \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$






$$\Delta P = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\frac{1}{R_1} \simeq \frac{1}{R_0 + \epsilon e^{\omega t + ikz}}$$

$$= \frac{1}{R_0\left[1 + \frac{\epsilon}{R_0} e^{\omega t + ikz}\right]}$$

$$= \frac{1}{R_0}\left[1 - \frac{\epsilon}{R_0} e^{\omega t + ikz} + \text{small terms}\right]$$

$$R = R_0 + \epsilon e^{\omega t + ikz}$$

$$\frac{1}{R_1} \approx \frac{1}{R_0} - \frac{\epsilon}{R_0^2} e^{\omega t + ikz}$$

$$\frac{1}{R_2} = \epsilon k^2 e^{\omega t + ikz}$$

$$\Delta P = \sigma\left(\frac{1}{R_0} - \frac{\epsilon}{R_0^2} e^{\omega t + ikz} + \epsilon k^2 e^{\omega t + ikz}\right)$$

$$\Delta P_{mean} + \Delta P_{pert.} = \sigma\left[\frac{1}{R_0} - \epsilon e^{\omega t + ikz}\left(\frac{1}{R_0^2} - k^2\right)\right]$$

$$\Delta P_{mean} = \frac{\sigma}{R_0}$$


$$\therefore \Delta P_{pert} = -\sigma \epsilon e^{\omega t + ikz}\left(\frac{1}{R_0^2} - k^2\right)$$

But $\Delta P_{pert}$ is same as $p(x)$ at $x = R$.

But $\Delta P_{out}$ is same as $p(r)$ ...

$$\Rightarrow \underbrace{P(r)\Big|_{r=R}}_{p(r) \text{ at } r=R} \cdot \cancel{e^{\omega t + ikz}} = -\sigma \epsilon \, \cancel{e^{\omega t + ikz}} \left( \frac{1}{R_0^2} - k^2 \right)$$

$$\Rightarrow \boxed{-\frac{\rho \omega C_1}{K} I_0(KR) = \frac{-\sigma \epsilon}{R_0^2} (1 - K^2 R_0^2)} \quad \text{(ii)}$$

Using (i), $C_1 = \dfrac{\epsilon \omega}{I_1(KR)}$

$$\Rightarrow \cancel{-}\frac{\rho \omega}{K} \cdot \frac{\cancel{\epsilon} \omega}{I_1(KR)} \cdot I_0(KR) = \cancel{-}\frac{\sigma \cancel{\epsilon}}{R_0^2} (1 - K^2 R_0^2)$$

$$\omega^2 = \frac{\sigma}{R_0^2} (1 - K^2 R_0^2) \cdot \frac{K}{\rho} \cdot \frac{I_1(KR)}{I_0(KR)} \times \frac{R_0}{R_0}$$

$$\Rightarrow \boxed{\omega^2 = \frac{\sigma}{\rho R_0^3} K R_0 (1 - K^2 R_0^2) \frac{I_1(KR_0)}{I_0(KR_0)}}$$

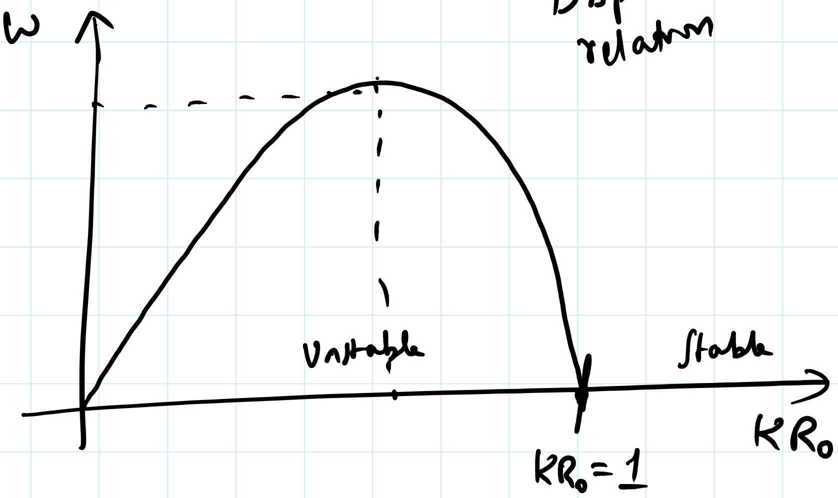
Dispersion relation

Instability occurs when $\omega > 0$. This happens only for $KR_0 < 1$.

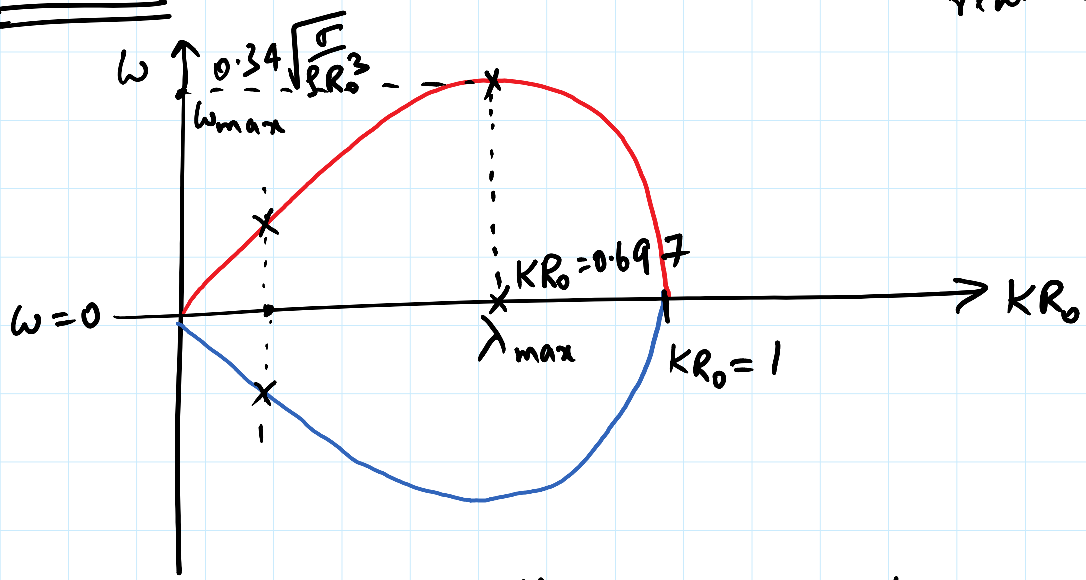
$$\Rightarrow \frac{2\pi}{\lambda} \cdot R_0 < 1 \quad \text{(or)} \quad \lambda > 2\pi R_0$$

$\omega$ ↑ Dispersion relation

Dispersion relation




**For $KR_0 < 1$:** $\omega = \pm\alpha$ when $\alpha$ is +ve real no.




for $\alpha > 0$; $e^{\omega t} = e^{\alpha t} \longrightarrow \infty$ as $t \to \infty$

for $\alpha < 0$; $e^{\omega t} = e^{-\alpha t} \longrightarrow 0$ as $t \to \infty$

**When $KR_0 > 1$:**

$\omega = \pm i\beta$ when $\beta > 0$

$$e^{\omega t} \nearrow e^{i\beta t} \longrightarrow \cos\beta t + i\sin\beta t$$

$$e^{\omega t} \searrow e^{-i\beta t} \longrightarrow \cos\beta t - i\sin\beta t$$

$\lambda_{max}$: Wavelength at maximum growth rate $\omega_{max}$

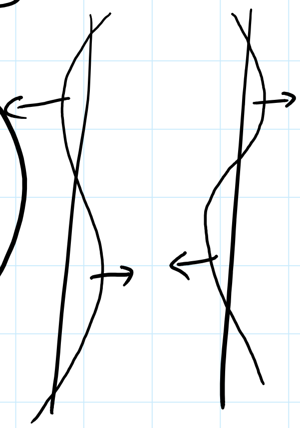
$$\lambda_{max} = \frac{2\pi}{k}\cdot\frac{R_0}{R_0} = \frac{2\pi}{kR_0}\cdot R_0$$

$$= \frac{2\pi}{0.697} R_0$$

$$\boxed{\lambda_{max} \approx 9.02\, R_0}$$

Rupture / Breakup time:-

$$t_{rupture} = \frac{1}{\omega_{max}} = 2.91\sqrt{\frac{\rho R_0^3}{\sigma}}$$



$$t \sim O(1) \times \sqrt{\frac{\rho R_0^3}{\sigma}}$$