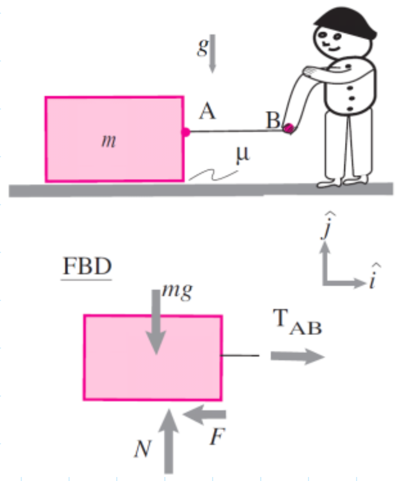
Equilibrium with Frictional Contact

We will only deal with Coulomb friction with a single coefficient of friction, μ, i.e.; $\mu_{static} = \mu_{dynamic}$ is the approximation we will use.



Dragging a block with friction:-

Force balance, from FBD, gives:

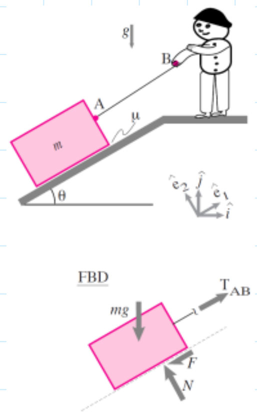
$$\sum \vec{F_i} = \vec{0} \Rightarrow -mg\,\hat{j} + T_{AB}\,\hat{i} + N\,\hat{j} - F\,\hat{i} = \vec{0} \quad \text{(a)}$$

We will use $F = \mu N$ (b)

Equations (a) & (b) are three equations for three unknowns: F, N & T_{AB}.

NOTE: for particle mechanics, we don't need to worry where $\vec{N}$ & $\vec{F}$ are applied.

Dragging a block on a ramp with friction:-



We can either

(i) Slide the block 'up' with rod AB

(ii) Push the block 'down' with rod AB

(iii) Hold it still.

Force balance:-

$$\sum \vec{f_i} = \vec{0}$$

$$\Rightarrow -mg\,\hat{j} + T_{AB}\,\hat{e}_1 + N\,\hat{e}_2 - F\,\hat{e}_1 = 0 \quad \text{(c)}$$

2 scalar equations.

Equation (c) along with friction relation gives us 3 equations for 3 unknowns in T_{AB}, F and N.

$$\{\text{Eqn } ©\} \cdot \hat{e}_1 = 0 \Rightarrow \quad -mg\sin\theta + T_{AB} - F = 0$$

$$\{\text{Eqn } ©\} \cdot \hat{e}_2 = 0 \Rightarrow \quad mg\cos\theta + N = 0$$

<u>Standard Coulomb friction model</u> :-

(i) $F = \mu N$ if the block is sliding up

(ii) $F = -\mu N$ if the block is sliding down

(iii) $-\mu N \le F \le \mu N$ if the block is not sliding

Solving eqn. © with the above frictional relations gives:

(i) $T_{AB} = mg(\mu\cos\theta + \sin\theta)$ if block is sliding up

(ii) $T_{AB} = mg(-\mu\cos\theta + \sin\theta)$ if the block is sliding down

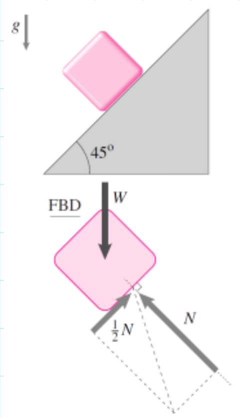
Note that if $\tan\theta < \mu$, then $T_{AB} < 0 \Rightarrow$ you'll have to push the block to slide it down

(iii) $mg(-\mu\cos\theta + \sin\theta) \le T_{AB} \le mg(\mu\cos\theta + \sin\theta)$ if the block is not sliding.

If $\tan\theta < 0$, then $T_{AB} = 0$ is a solution for no sliding, i.e; the block sits still on the ramp without pulling on the rod.

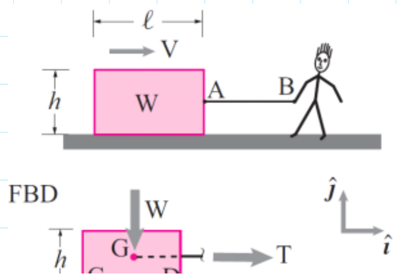
<u>No statics solution</u> :

A block on a ramp with angle 45° & with $\mu = 0.5$ cannot be in static equilibrium!



<u>Block as an extended body</u> :-

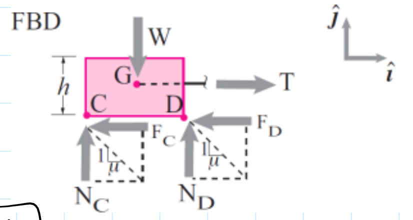
Does dragging cause an uneven force on the block?



FBD:

block?

From FBD:



$$N_C + N_D = W$$

and $\underbrace{F_C}_{\mu N_C} + \underbrace{F_D}_{\mu N_D} = T_{AB} \implies \boxed{T_{AB} = \mu W}$

Taking moment about C gives:

$$N_D = \frac{W}{2} + \frac{\mu h W}{2l} \quad \text{and} \quad N_D = \frac{W}{2} - \frac{\mu h W}{2l}$$

$\implies$ There is more pressure at D than at C.

Conditional Contact, Consistency & Contradictions:-

Problems involving 'contact' tend to conditional in nature:-

Ex: (i) Ground pushes up on the object to prevent interpenetration, if the pushing is positive, else the ground does not push up the body

(ii) Force of friction $= \mu N$ if there is slip, else the force is $< \mu N$.

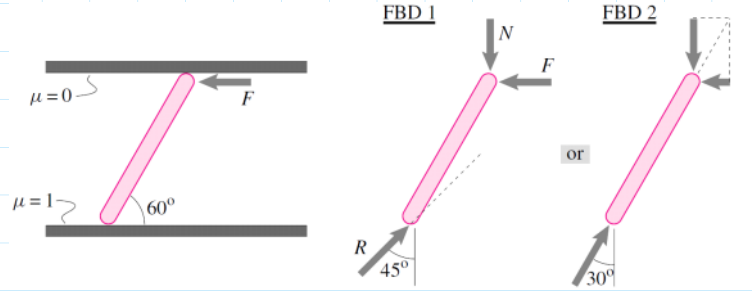
(iii) Distance between two points is kept from increasing if tension in the string is positive, else tension is zero.

Rules of thumb:-

On a FBD at every point of frictional contact

- *If* the direction of slip or impending slip is known, *either*
  - Draw a normal force N and a friction force $F = \mu N$ opposing the relative slip, *or*
  - Draw a single force R at an angle ϕ from the normal of the contact in the direction which resists slip (with $\tan\phi = \mu$)
- *If* there is no slip, *either*
  - Draw a normal force N and tangential force F *or*
  - Draw a single force vector $\vec{R}$ with unknown components
- *If* you don't know whether of not there is slip, *first*
  - Guess that there is no slip, *then*
  - Solve the equilibrium equations, *then*
    - *If* $F \leq \mu N$: you guessed right and have found a solution to both the equilibrium and friction equations.
    - *If* $F > \mu N$: you guessed wrong and have to guess that there is slip in one direction (guess which), *then*
      - see if you can solve the equilibrium equations, if not *then*
      - assume slip in the opposite direction and try to solve the equilibrium equations, if you can't, *then*
      - the problem has no solution

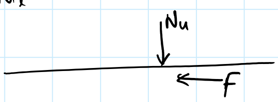
Multiple solutions with problems involving friction:-



Consider a weightless rod which is just long enough to make 60° angle with the walls of a channel. Lower wall has $\mu = 1$ and upper wall is frictionless.
What is the force needed to keep it in equilibrium?

If the rod is assumed to be sliding, we get FBD 1.
The forces shown can be in equilibrium if all the forces are zero. So a solution is that the rod slides in equilibrium with no force.

If the rod is not sliding, the friction force on the lower wall can be at any arbitrary angle since we require
$|F_l| \leq \mu N_l = N_l \Rightarrow -N_l \leq F_l \leq N_l$ where
N_l & F_l are reaction and frictional forces on the lower wall.



N_u : Reaction at upper wall.

For equilibrium, we have $N_l = N_u$
& $F = F_l$.

The friction force on the lower wall can

$4 \quad F = f_2.$

The friction force on the lower wall can be at any angle between $\pm 45°$.

We can equilibrium in $\underline{\underline{FBD\,2}}$ for arbitrary positive F.

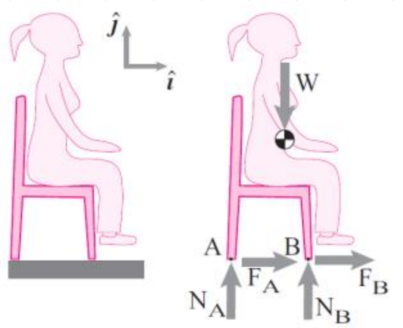
Summary: We can either have freely slipping solutions with no force or jammed (stuck) solutions with arbitrary force.

This physically corresponds to one being able to easily slide a rod like this down a slot and then also at a different instant, have the rod totally jam. Such "self-locking" rods are used in some rock-climbing equipment.

Statically indeterminate problems:-

When two or more points of contact have friction, then statical indeterminacy is likely if there is no slip.

Ex: Person sitting on chair:



$$|F_A| \leq \mu N_A$$

$$4 \quad |F_B| \leq \mu N_B$$

Equilibrium:

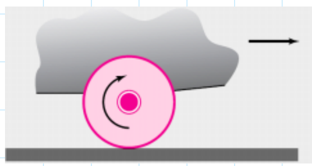
$$f_A + f_B = 0$$

$$4 \quad N_A = N_B = \frac{W}{2} \qquad \text{(Assume } W \text{ is in the middle)}$$

All we can say is

$$-\frac{W}{2} \leq f_A \leq \frac{W}{2} \qquad 4 \qquad f_B = -f_A$$

Wheel as a two-force body:

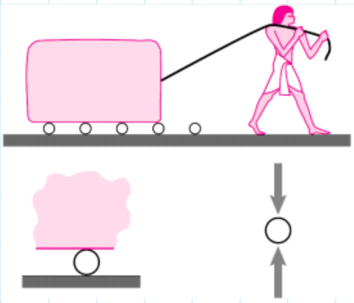


Without a wheel, it takes a force of approximately 'μW' to drag something weight W.

For teflon, $\mu \approx 0.1$ 4 for stone on ground, $\mu \approx 0.6$ and, $\mu \approx 1$ for stone on ground.

One way to reduce friction is to have rolling logs below the cart.

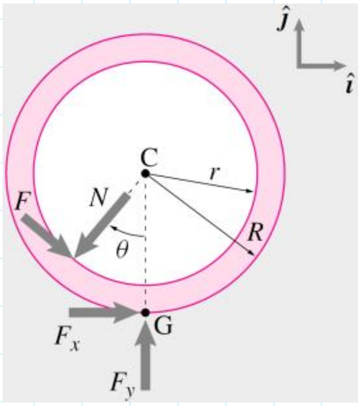
This idea has a deficiency – logs have



This idea has a deficiency – logs have to moved from back to the front of the boulder.

One way to avoid this is to introduce a fixed non-rotating shaft to a circular disk with a hole – a wheel.

<u>free-body diagram of the wheel</u>:-



<u>Assumption</u>: Wheel weight is assumed negligible.

r: Radius of bearing (shown here to be very large)

R: Radius of the wheel.

The force of the axle on the wheel has a normal component N and a frictional component F. The forces due to the ground are F_x and F_y.

Unknowns: N, F, θ, F_x.

F_y is determined by the weight of the cart.

<u>Equations</u>:- $F = \mu N$ (frictional relation)

<u>force balance</u>:- $F_x \hat{i} + F_y \hat{j} + N(-\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) + F(\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) = 0$

<u>Moment about C</u>:- $Fr + F_x R = 0$

We are mainly interested in finding the resultant force <u>F_x</u> on the ground.

We can solve the four scalar equations for the four unknowns $F_x, N, F,$ and θ in terms of r, R, F_y and μ.

<u>Equations</u>:

$$F = \mu N \quad — (1)$$

$$F_x - N\sin\theta + F\cos\theta = 0 \quad — (2)$$

$$F_y - N\cos\theta - F\sin\theta = 0 \quad — (3)$$

$$Fr + F_x R = 0 \quad — (4)$$

from (1), (2): $F_x - N\sin\theta + \mu N\cos\theta = 0 \quad — (5)$

from (1) & (3): $F_y - N\cos\theta - \mu N\sin\theta = 0 \quad — (6)$

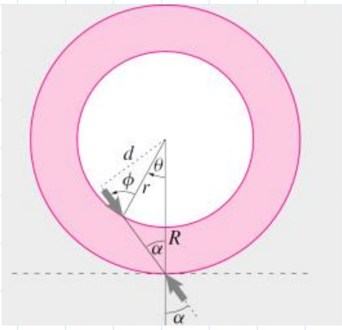
from (6): $F_y = N(\cos\theta + \mu\sin\theta) \quad — (7)$

(2) × $\cos\theta$ − (3) × $\sin\theta$: $F_x\cos\theta - F_y\sin\theta - \cancel{N\sin\theta\cos\theta} + \cancel{N\cos\theta\sin\theta}$
$+ F\cos\theta\cdot\cos\theta + F\sin\theta\cdot\sin\theta = 0$

$$\Rightarrow \quad F_x \cos\theta - F_y \sin\theta + F = 0 \quad — (8)$$

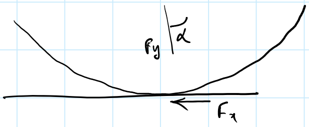
We can follow a slightly different approach to obtain μ_{eff}!

The wheel is really a two-force body. So for equilibrium, we require two equal & opposite collinear forces at the two contact points as shown.



ϕ : Friction angle, i.e; $\mu = \tan\phi$
where μ is coeff. of friction between axle and wheel.

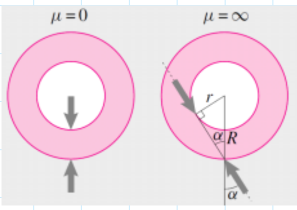
α can be thought of as a friction angle due to an "effective" friction at the wheel.



Note that α is not the sliding friction coeff. between wheel & ground, since we assume that the wheel is in pure rolling motion.

Instead, $\mu_{eff} = \tan\alpha$ is the specific resistance or the coefficient of rolling friction.
If there was no wheel, $\mu_{eff} = \mu$.

<u>Extreme</u> <u>limits</u>: $\mu = 0$ and $\mu = \infty$.



$\mu = 0$: Frictionless bearing $\Rightarrow \phi = 0^\circ$
No ground resistance.

$\mu = \infty$: Infinite friction coeff $\Rightarrow \phi = 90^\circ$.
Here we get $\sin\alpha = \dfrac{r}{R}$

Thus if an axle has a diameter of 10 cm & a wheel of 1m, then $\sin\alpha = 0.1/1$. For such small values of $\sin\alpha$, we can take $\tan\alpha \approx \sin\alpha \approx \mu_{eff}$.,

From the two-force body FBD, we have

$$\underbrace{r\sin\phi}_{d} = \underbrace{R\sin\alpha}_{d}$$

$$\Rightarrow \quad \sin\alpha = \frac{r}{}$$

Limiting cases:

$$\phi = 0 \Rightarrow \sin\alpha = 0 \Rightarrow \tan\alpha = 0 \Rightarrow \mu_{eff} = 0$$

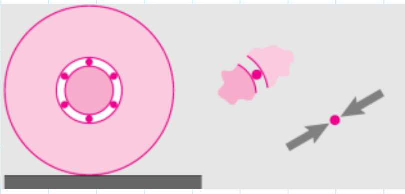
$$\phi = 90^\circ \Rightarrow \sin\phi = 1 \Rightarrow \sin\alpha = \frac{r}{R}$$

$$\Rightarrow \tan\alpha = \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}} = \frac{r}{R}\frac{1}{\sqrt{1-\frac{r^2}{R^2}}} = \frac{r}{\sqrt{R^2-r^2}}$$

If $r << R$, & μ is small, $\sin\phi \approx \tan\phi = \mu$

& $\sin\alpha \approx \tan\alpha = \boxed{\mu_{eff} = \frac{r}{R}\cdot\mu}$

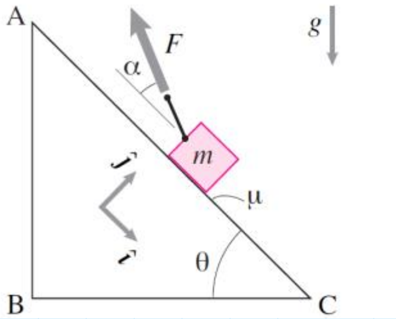
We can now combine the wheel with the rolling logs "inside" to get a ball-bearing wheel.



Each ball is a two-force body and thus only transmits radial loads.

Real ball bearings are not perfectly smooth nor perfectly rigid, so to obtain small μ_{eff}, we should ideally keep $\frac{r}{R} << 1$.

Example: Ramp on block sliding up or down:-



$m = 10$ kg

Block pushed up by force F.

$\mu = 0.7$

① Let $\theta = 60^\circ$ & $\alpha = 0^\circ$. If block slides down, what is the tension in the string.

② Let $\theta = 30^\circ$ & $\alpha = 30^\circ$. If $F = 20N$, what is the friction on the block?

③ Let $\theta = 60^\circ$ & $\alpha = 30^\circ$. If $F = 10N$, what is the friction on the block?

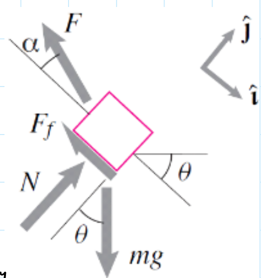
④ For $\theta = 30^\circ$ & $\alpha = 30^\circ$, what will be the tension in the string to make the block just slide up on the slope? Express your answer in terms of the weight of the block.

Solution: $\sum \vec{F} = \vec{0}$

$$\Rightarrow \vec{F} + \vec{N} + \vec{F_f} + \vec{W} = 0$$

$$\Rightarrow F(-\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}) + N\hat{j} \quad \hat{i}$$

$$+ mg(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) = \vec{0} \quad ①$$



① Block sliding down:- If sliding rate is steady or very slow, then $\vec{a} \approx 0$. Also, $\vec{F_f} = -\mu N\,\hat{i}$

from ①, we have :
$$-F\cos\alpha - \mu N + mg\sin\theta = 0$$
$$F\sin\alpha + N - mg\cos\theta = 0$$

Eliminating N, we have
$$F = \left(\frac{\sin\theta - \mu\cos\theta}{\cos\alpha - \mu\sin\alpha}\right) mg$$

Substituting the numerical values, we get $\boxed{\vec{F} = (-51N)\hat{i}}$

② Block sliding or not sliding - not known :-

$F = 20N$ & $\alpha = 30°$, $\theta = 30°$

Since we do not know if the block is sliding or not. Let us assume static equilibrium in the given configuration & solve for the friction force F_f. We will then check if it satisfies friction law for static equilibrium ($|F| \le \mu N$).

Substituting $\vec{F_f} = -F_f\,\hat{i}$, we get

$$\left.\begin{aligned} -F\cos\alpha - F_f + mg\sin\theta &= 0 \\ F\sin\alpha + N - mg\cos\theta &= 0 \end{aligned}\right\} \quad F, \alpha, \theta, mg \text{ are known.}$$

$F_f = mg\sin\theta - F\cos\alpha$
& $N = mg\cos\theta - F\sin\alpha$

Putting in the numbers, $F_f = 31.73N$
& $N = 74.96N$.

Maximum possible value of friction force is $\mu N = 0.7 \times 74.96N$
$= 52.47N$.

Thus $|F_f| < \mu N$, and therefore, our assumption of static equilibrium is valid.

Thus $\vec{F} = -31.73N\,\hat{i}$

③ Block sliding or not sliding - not known, again :-

$F = 10N$, $\alpha = 30°$ & $\theta = 60°$

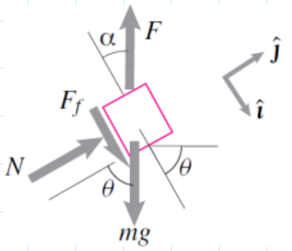
Again, assuming static equilibrium, we get
$F_f = 76.3N$ & $N = 44N$

Here $|F_f| > \mu N$. Clearly, F_f is not less than or equal to μN, & therefore our assumption of static equilibrium is not valid. In fact, for the given parameters, the block will accelerate downhill - a problem of dynamics.

However, the friction force remains constant, at its maximum value of $\vec{F_f} = \mu N = 33.88N$ once the sliding starts, accelerating or not.

$\vec{F_f} = -33.88N\,\hat{i}$

(4) Block just about to slide upwards:-



Now $F_f = \mu N$.

$$-F\cos\alpha + \mu N + mg\sin\theta = 0$$

$$F\sin\alpha + N - mg\cos\theta = 0$$

Eliminating N, we get

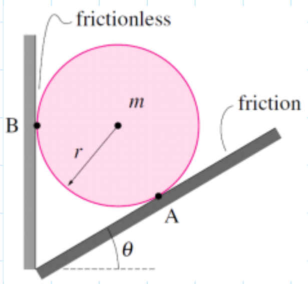
$$F = \left(\frac{\sin\theta + \mu\cos\theta}{\cos\alpha + \mu\sin\alpha}\right) mg$$

$$= \left(\frac{1.216}{1.216}\right) mg = mg.$$

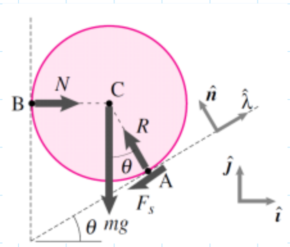
Does the answer makes sense? Yes, it does.
For given θ & α, the string tension is vertical. If it balances the weight of the block, normal force goes to zero & so does the friction force.
If tension increases by a tiny amount, the block tends to slide up.

Example:- How much friction does the cylinder need?

Find the friction force on the cylinder from the incline.



Free-Body diagram:-



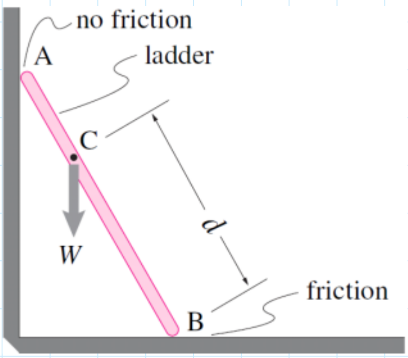
We need to find F_s.

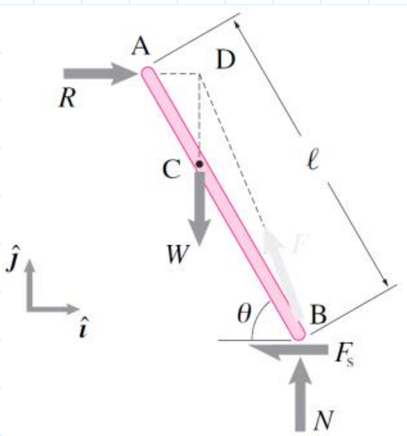
$$\sum \vec{M}_c = 0 \Rightarrow r F_s(-\hat{k}) = 0$$

$$\Rightarrow F_s = 0$$

Force of friction on the cylinder is zero.
Infact F_s is independent of θ.

Note: The cylinder is a 3-force body (two contact forces at A & B and one gravity force).


no friction
A
ladder
C
d
W
B
friction


A
D
R
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