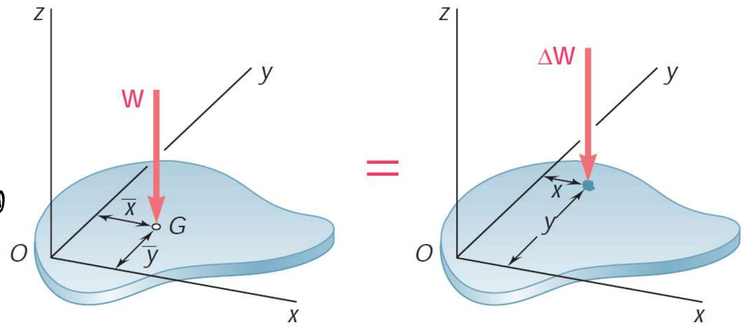
Characterising two-dimensional surfaces

Centre of gravity of a 2D Body

Consider a flat plate of known thickness.



If there is a distributed load W acting on it, it may be useful to sometimes replace it with a point load acting at its centre of gravity, G.

We can divided the plate into 'n' tiny elements. Each element then has a load $\Delta W_1, \Delta W_2, \Delta W_3, \ldots$ and so on.

Total force, $W = \Delta W_1 + \Delta W_2 + \Delta W_3 + \ldots + \Delta W_n$

To find the coordinates of G where W should be applied, we equate the moment of W about x & y axes with the moments of individual elemental loads:

$$\sum M_y : \quad \bar{x} W = x_1 \Delta W_1 + x_2 \Delta W_2 + \ldots + x_n \Delta W_n$$

$$\sum M_x : \quad \bar{y} W = y_1 \Delta W_1 + y_2 \Delta W_2 + \ldots + y_n \Delta W_n$$

For large n, we can now write:

$$\boxed{\begin{aligned} W &= \int dW \\ \bar{x} W &= \int x \, dW \\ \bar{y} W &= \int y \, dW \end{aligned}}$$

first moments

The same thing can be done for a wire or cable. In most cases, the CG of such objects does not coincide with the object.

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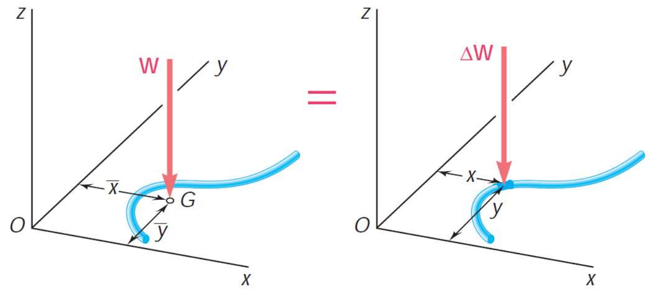
cases, the cm of
object.



Centroids of Areas & Lines:-

For a flat homogeneous plate, weight of the plate, we know

$$\Delta W = g \cdot t \, \Delta A$$

$$\& \qquad W = g t A.$$

Substituting ΔW & W in earlier eqns. & dividing by gt, we have:

$$\boxed{\begin{aligned} A &= \int dA \\ \bar{x} A &= \int x \, dA \\ \bar{y} A &= \int y \, dA \end{aligned}}$$

Here $\int x\,dA$ is called the first moment of area about the y-axis & is denoted by Q_y. Similarly Q_x is first moment of area bout x-axis.

$$\boxed{\begin{aligned} Q_y &= \int x \, dA = \bar{x} A \\ Q_x &= \int y \, dA = \bar{y} A \end{aligned}}$$

And for a wire or cable.

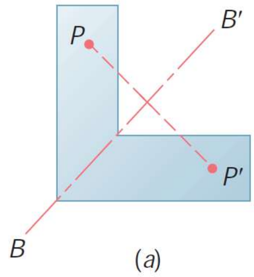
$$\Delta W = g \cdot a \, \Delta L$$

$\hookrightarrow$ cross-sectional area of the wire

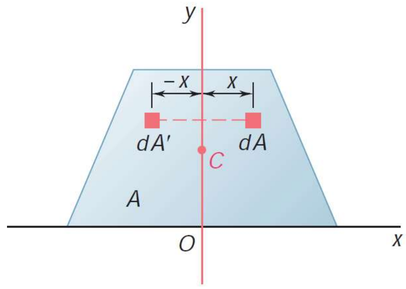
$$\Rightarrow \quad \bar{x} L = \int x \, dL \quad \& \quad \bar{y} L = \int y \, dL$$

Some useful Symmetries to note :-

① An area is said to be symmetric about an axis BB' if for every point P, there exists a point P' on the area such that PP' ⊥ BB'.



When an area possesses an axis of symmetry, its first moment w.r.to that axis is zero.



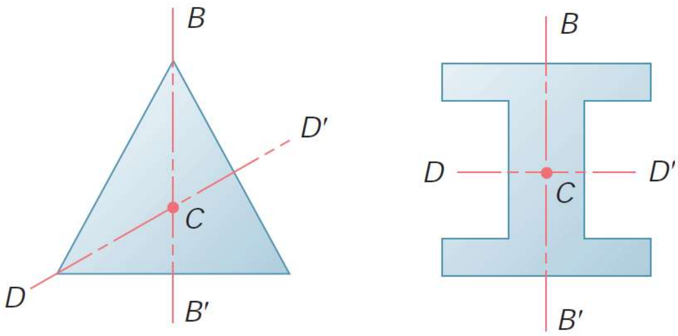
Ex: The y-axis of this area is the line of symmetry.

$\Rightarrow$ first moment about y-axis, i.e. $Q_y = 0$.

Thus $\bar{x} = 0$

Therefore, if an area possesses a line of symmetry, its centroid has to lie on that line.

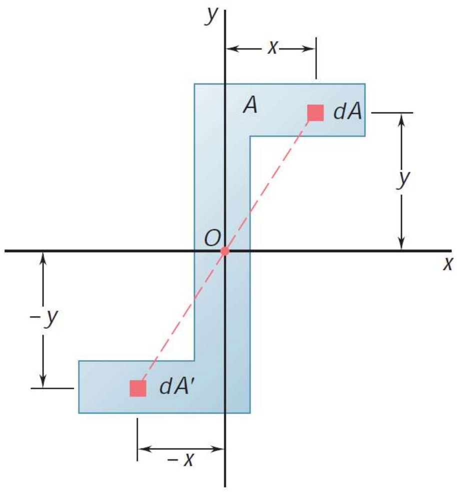
② If an area has two lines of symmetry, its centroid lies at the intersection of these lines.



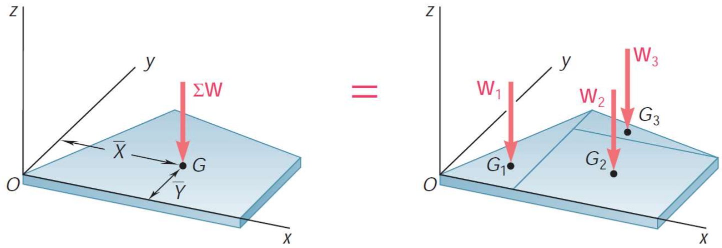
③ An area is said to be symmetric with respect to centre O if for every element of area ΔA of coordinates x & y, there exists an element dA' of equal area with coordinates −x & −y. It then follows that both $Q_x = Q_y = 0$

$\Rightarrow \quad \bar{x} = \bar{y} = 0,$

i.e.; centroid coincides with centre of symmetry O.

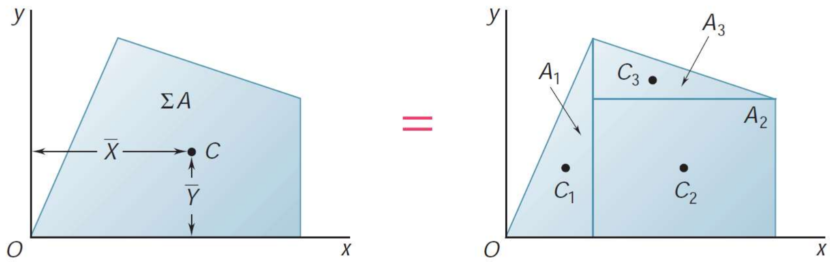


Composite plates :-



$$\Sigma M_y : \quad \bar{X}(W_1 + W_2 + \dots W_n) = \bar{x}_1 W_1 + \bar{x}_2 W_2 + \dots + \bar{x}_n W_n$$

$$\Sigma M_x : \quad \bar{Y}(W_1 + W_2 + \dots W_n) = \bar{y}_1 W_1 + \bar{y}_2 W_2 + \dots + \bar{y}_n W_n$$



Similarly:

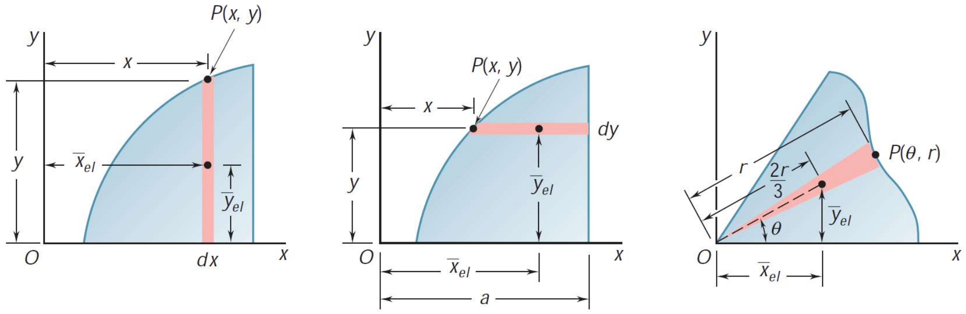
$$Q_y = \bar{X} \, \Sigma A = \Sigma \bar{x} A$$

$$Q_x = \bar{Y} \, \Sigma A = \Sigma \bar{y} A$$

Determining centroids by integration :-

$$Q_y = \bar{x} A = \int x \, dA = \int \bar{x}_{el} \, dA$$

$$Q_x = \bar{y}A = \int y\, dA = \int \bar{y}_{el}\, dA$$



$\bar{x}_{el} = x$

$\bar{y}_{el} = \frac{y}{2}$

$dA = y\,dx$

$\bar{x}_{el} = \frac{a+x}{2}$

$\bar{y}_{el} = y$

$dA = (a-x)\,dy$

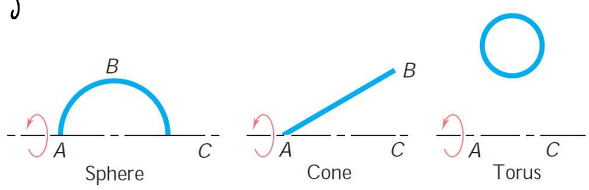
$\bar{x}_{el} = \frac{2r}{3}\cos\theta$

$\bar{y}_{el} = \frac{2r}{3}\sin\theta$

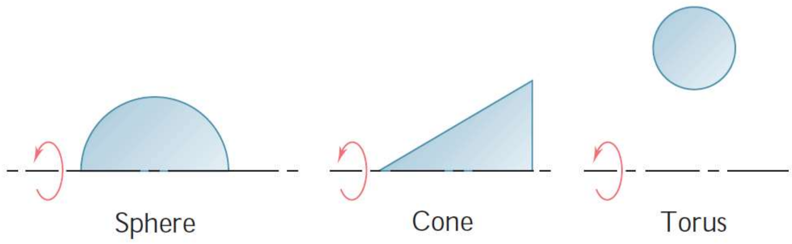
$dA = \frac{1}{2} r^2\, d\theta$

Theorems of Pappus - Guldinus:-

A surface of revolution is a surface generated by rotating a curve.

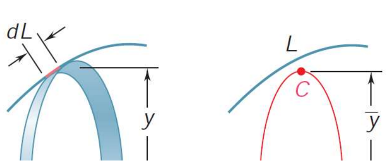


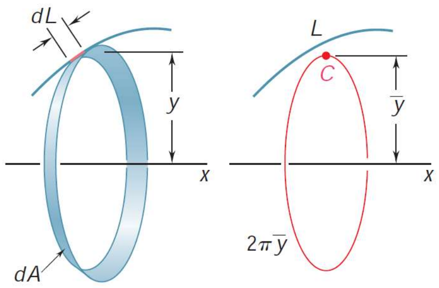
Similarly, we can generate a "body of revolution".



Theorem: The area of a surface of revolution is equal to the
length of the generating curve times the distance traveled by
the
centroid of the curve while the surface is being generated.

Proof:





Area generated by line of length dL is $dA = 2\pi y\, dL$

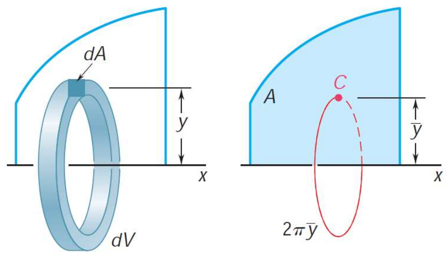
Entire area generated by L = $A = \int 2\pi y\, dL$

But $\int y\, dL = \bar{y} L$

$\Rightarrow \quad A = 2\pi \bar{y} L$

Body of revolution:-

Theorem II: The volume of a body of revolution is equal to the generating area times the distance traveled by the centroid of the area while the body is being generated.



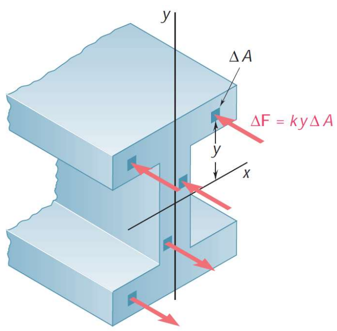
Proof: Consider an element of area dA.

The volume dv generated by revolving about x-axis is given by $dv = 2\pi y\, dA$

$$V = \int dv = \int 2\pi y\, dA = 2\pi \int y\, dA = 2\pi \bar{y} A$$

Second moment of area:-

For a beam in pure bending, the internal forces in any section of the beam are distributed forces whose magnitude is given by



ΔA

$$\Delta F \propto y \Delta A$$

or $\Delta F = k y \Delta A$

where y is the distance of the element area from an axis passing through the centroid.
↳ the neutral axis.

Net resultant force $\vec{R}$ of the elemental forces ΔF over the section is

$$R = \int k y \, dA = k \int y \, dA$$

But $\int y \, dA = \bar{y} A = Q_x = 0$ since we are measuring y about centroidal axis i.e., $\bar{y} = \bar{x} = 0$

$\Rightarrow R = 0 \Rightarrow$ The system of forces ΔF reduces to a couple.

The magnitude of this couple (bending moment) must be equal to the sum of the moments:

$$\Delta M_x = y \Delta F = k y^2 \Delta A$$

Integrating, $M = \int k y^2 \, dA = k \underbrace{\int y^2 \, dA}_{\text{second moment of area.}}$