Instabilities due a vortex at a density interface: gravitational and centrifugal effects

Harish N Dixit and Rama Govindarajan

Abstract A vortex placed at an initially straight density interface winds it into an ever-tightening spiral. This flow then displays rich dynamics, due to inertial effects caused by density stratification (non-Boussinesq effects), and gravitational effects. In the absence of gravity we showed recently that the flow is subject to centrifugal Rayleigh-Taylor and spiral Kelvin-Helmholtz instabilities. The latter grows slightly faster than exponentially. In this paper we present computations including gravity with and without and with inertial effects. Gravity modifies the spiralling process and contributes to the breakdown of the vortex. When both effects are allowed to operate together, the resulting flow has a complex radial character, with small-scale structures near the vortex core attributed to non-Boussinesq effects, and large scale roll-up due to gravity followed by breakdown.

1 Introduction

Vortical structures in stratified flows display a range of interesting instabilities and non-monotonic behaviour, see e.g. [1] and [2]. The present study is two-dimensional, of a lone vortex with its axis perpendicular to the plane of density stratification, with and without gravity. An initially flat density interface is wound up into an increasingly tightened spiral by the vortex, similar to how it would advect a patch of passive scalar (see for example [5]). In the absence of gravity, centrifugal forces are predominant, and we showed recently [4] that two kinds of instabilities, of a centrifugal Rayleigh-Taylor (CRT) and spiral Kelvin-Helmholtz (SKH) types are

Harish N Dixit
Engineering Mechanics Unit, Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore 560064, INDIA, e-mail: hdixit@jncasr.ac.in

Rama Govindarajan
Engineering Mechanics Unit, Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore 560064, INDIA e-mail: rama@jncasr.ac.in

triggered. The former arises from a mechanism similar to the Rayleigh-Taylor instability of a vortex with a heavy core, as studied by [9] and [6]. The latter arises purely from the fact that the density interface, being spiral, is not quite circular. Both instabilities would be missed upon making a passive-scalar approximation, i.e., upon neglecting the inertial effects of density stratification. Gravity is unnecessary in this process, but since the density stratification is stable with respect to gravity to begin with, it would be interesting to see the effect that gravity would have on these instabilities, and on the resulting breakdown into a possibly turbulence-like state. We show that gravity has a profound effect on the dynamics and hastens the final breakdown into a turbulence-like state. We present simulations (i) including centrifugal effects (non-Boussinesq) but without gravity, and (ii) gravity present, under the Boussinesq approximation, and (iii) of the two combined.

The vorticity and density balance equations are given in the inviscid, infinite Peclet number limit by

$$\rho \frac{\mathscr{D}\Omega}{\mathscr{D}t} = -\nabla\rho \times \frac{\mathscr{D}\mathbf{u}}{\mathscr{D}t}. - g\frac{\partial \rho}{\partial x} \tag{1}$$

$$\frac{\mathscr{D}\rho}{\mathscr{D}t} = 0, \tag{2}$$

where $\mathscr{D}/\mathscr{D}t \equiv \partial/\partial t + \mathbf{u}\cdot\nabla$, and $\mathbf{u} = (u_r\mathbf{e}_r, u_\theta\mathbf{e}_\theta)$, Ω, ρ and g denote the velocity vector (of radial and azimuthal components), the vorticity, the density and gravity respectively. The flow is taken to be incompressible, so $\nabla\cdot\mathbf{u} = \mathbf{0}$. The two cases we present in sections 2 and 3 correspond to the neglect of the second and the first term on the right hand side of (1) respectively, while the full equations are solved in the third case.

2 Centrifugal effects

A brief description of the flow is given here, further details are available in [4]. Consider a point vortex of circulation $2\pi\Gamma$ located at an initially horizontal density interface, with a difference $\Delta\rho$ in density across it. The point vortex causes a spiralling of the density interface, as in figure 1.

For large time or small radius, the spacing between successive turns of the spiral scales as

$$\lambda \sim \frac{r^3}{\Gamma t}. \tag{3}$$

When a small amount of diffusivity, κ is present in the flow, the prominent length-scales in the flow can be expressed in terms of the Peclet number, $Pe = \Gamma/\kappa$, of the flow, as shown in fig.(1). If we assume that these jumps are circular, then the flow is analogous to the planar Rayleigh-Taylor instability, except that the gravitational potential arises from the centrifugal forces. Density jumps from ρ_h to ρ_l corresponding to heavy and light fluids. Just as a heavy fluid above a light one is unstable due

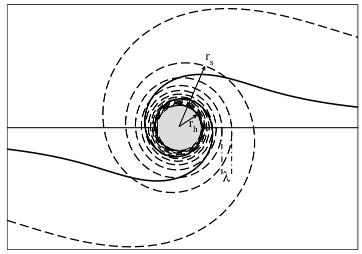


Fig. 1 Evolution with time of an initially horizontal density interface due to a point vortex at the origin [4]. The dashed line is at a later time than the solid line. At finite Peclet number, a central region indicated by a grey circle is homogeneous, and the spiral extends upto a length $r_s \sim l_d Pe^{1/2}$. At large Pe and large time many density jumps exist.

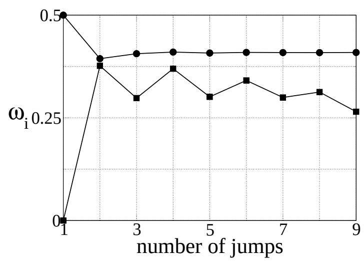


Fig. 2 Maximum growth rate ω_i of disturbance as a function of the number of density jumps for $m = 5$ and $\mathscr{A} = 0.05$. The circles show the growth rate with the core fluid being heavy, at $\rho_h = 1.05$, while squares are for a light core, at $\rho_l = 0.95$. The first two jumps are located at (a) $r_1 = 0.1, r_2 = 0.102$ (b) $r_1 = 0.1, r_2 = 0.105$, with the remaining jumps spaced out as r^3, in accordance with the spacing λ between the jumps. The growth rate has been normalised by Γ/r_1^2.

to gravity, a heavy fluid inside a light one is unstable to centrifugal forces, here we refer to this as the CRT instability. It is shown analytically for inviscid flow with $\kappa = 0$ that although stabilising and destabilising density jumps occur alternately, the net effect is destabilising, even in the case of a light core, see fig.(2).

We have so far approximated the interface to be circular, when in reality, the interface is actually in the form of a spiral. Baroclinic torque is created when a density interface is not strictly perpendicular to centrifugal acceleration, and this torque produces vorticity here since the interface is not perfectly circular.

For a point vortex, any passive interface around it would take the form of a Lituus spiral, which can be represented parametrically as

$$\theta_s = \frac{\Gamma t}{r^2}. \tag{4}$$

From equation (4) we may obtain the angle α between the spiral and a circle, crossing each other at a given point and sharing the same origin and radius as

$$\tan\alpha = \frac{r^2}{2\Gamma t}, \tag{5}$$

so the assumption of a circular interface is better at smaller radii or late times.

Returning to the vorticity equation (1), neglecting gravity, and assuming that the effect of the circulation Γ of the point vortex is far greater on the basic flow than of that which is newly created, we may write

$$\frac{\mathscr{D}\Omega}{\mathscr{D}t} \simeq -\frac{\nabla\rho}{\rho} \times \left[\frac{U^2}{r}\mathbf{e}_r\right]. \qquad (6)$$

At high θ_s, i.e., at large time at a given radius, the integral of the above may be approximated as

$$\Delta U_\theta \simeq \mp \mathscr{A} U \log(\frac{2\Gamma t}{r^2}). \qquad (7)$$

At every unstable density jump, negative vorticity is created and vice versa, resulting in heavier fluid travelling faster and lighter fluid slower. A spiral Kelvin-Helmholtz (SKH) instability thus ensues, and combines with the centrifugal Rayleigh-Taylor (CRT) instability.

Given that ΔU_θ increases logarithmically in time, we have a spiral Kelvin-Helmholtz instability with a slightly faster than exponential growth rate, i.e., $u_r \sim at^{bt}$ where a and b are constants.

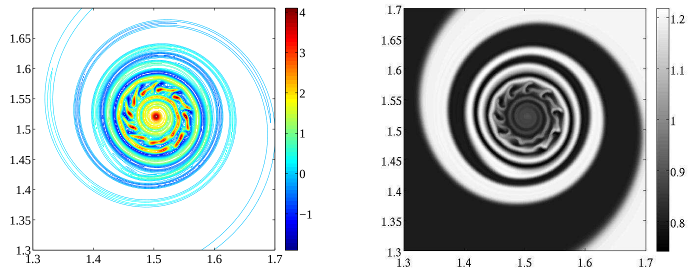
While a perfectly sharp density interface is increasingly unstable to increasing wavenumber, a small but finite thickness of the interface determines that the fastest growing wave has a wavelength comparable to the thickness, as expected. The replacement of a point vortex by a Rankine with smoothed-out edges does not change the answer qualitatively either, except within the vortex core.

Numerical simulations are obtained as follows. The 2D Navier-Stokes equations including non-Boussinesq effects are solved using the Fourier pseudospectral method in space. Inviscid (with hyperviscosity) results are presented here but viscous results are qualitatively no different. Figure 3 shows the vorticity and density fields including non-Boussinesq effects in the absence of gravity. Vorticity of alternating sign is produced in the form of two interwoven spirals along the density interfaces, and instabilities ensue, consistent with our theoretical predictions.

3 Gravitational effects

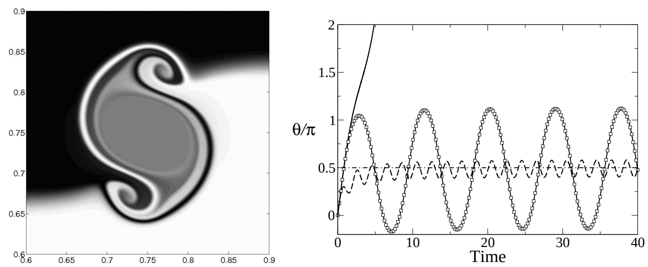
We now consider the effects of gravitational effects alone and employ the Boussinesq approximation. These are the first results from an ongoing study [3]. In the linear framework, such a flow has been treated in [7].

Figure 4(a) shows vorticity and density contours from simulations when the Froude number is 4.6. Centrifugal effects are not included, due to which the spiralling process of the interface is impeded. In general, gravity becomes important when $g > \Gamma^2/r^3$. An estimate of this ratio for realistic trailing vortices was given by [8].



(a) (b)

Fig. 3 Vorticity (a) and density (b) fields in the inviscid simulations without gravity. The time is 157 times the period of rotation of the vortex core, and the Atwood number is 0.2.



(a) (b)

Fig. 4 (a) Density field in the inviscid simulations using the Boussinesq approximation. The time is 157 times the period of rotation of the vortex core, and the Atwood number is 0.2. (b) Evolution of angle made by the satellite vortices with time for various values of gravity.

The inclusion of gravity results in the formation of two satellite vortices, whereas the non-Boussinesq equations (without gravity) results in the formation of a small scale instability. When the two effects are combined, features of both the cases can be seen as shown in fig.(5). The small scale non-Boussinesq effects are seen near the vortex center, whereas the large scale overturning occurs further away from the center. More details on these effects can be seen in [3].

4 Conclusions and outlook

We have shown that interesting dynamics emerges when the passive-scalar approximation is not made, as usually done for small Atwood number flows. When density variation is very sharp, gradients in density can lead to significant barloclinic torque, making the Boussinesq equations incomplete. Gravity impedes the original spiralling process, but gives rise to satellite spirals instead. Both speed up the de-

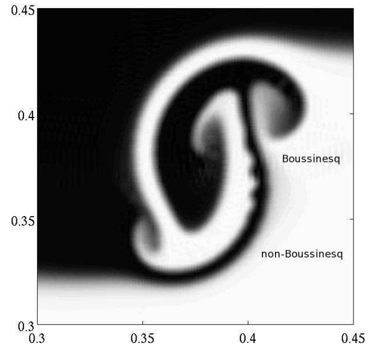


Fig. 5 Density field in a full simulation combining effects of gravity and inertia related density effects. Small scale undulation near the center of the vortex is due non-Boussinesq terms, and large scale overturning seen at an angle of 60^0 is due to gravitational effects.

struction of the identity of the vortex, and result in a turbulence-like state. More work is in progress and will appear in a future study ([3]). Another area of interest would be influence of these instabilities in multiple vortex scenarios, as in trailing vortices and vortex merger problems.

References

1. P. Billant and J. M. Chomaz. (2000) Experimental evidence for a new instability of a vertical columnar vortex pair in a strongly stratified fluid. J. Fluid Mech. 418, pp.167-188.
2. L. K. Brandt and K. K. Nomura. (2007) The physics of vortex merger and the effects of ambient stable stratification. J. Fluid Mech. 592, pp.413-446.
3. Harish N Dixit and Rama Govindarajan. Vortex in a density stratified flow: gravitational and centrifugal effects. *Preprint*
4. Harish N Dixit and Rama Govindarajan. (2009) Vortex-induced instabilities and accelerated collapse due to inertial effects of density stratification. *Submitted to J. Fluid Mech.*, and movies at http://www.jncasr.ac.in/rama/rg_research.htm
5. P. Flohr and J. C. Vassilicos. (1997) Accelerated scalar dissipation in a vortex. J. Fluid Mech. 348, pp.295-317.
6. L. Joly, J. Fontane and P. Chassaing. (2005) The RayleighTaylor instability of two-dimensional high-density vortices. J. Fluid Mech. 537, pp.415-431.
7. J. C. S. Meng and J. W. Rottman. (1988) Linear internal waves generated by density and velocity perturbations in a linearly stratified fluid. J. Fluid Mech. 186, pp.419-444.
8. P. Orlandi, G. F. Carnevale, S. K. Lele and K. Shariff. (1998) DNS study of stability of trailing vortices. CTR, Proc. Summer Program.
9. D. Sipp, D. Fabre, S. Michelin and L. Jacquin. (2005) Stability of a vortex with a heavy core. J. Fluid Mech. 526, 67-76.