Performance Analysis of Maximum-Likelihood Multiuser Detection in Space-Time Coded CDMA with Imperfect Channel Estimation*

G. V. V. Sharma and A. Chockalingam, *Senior Member, IEEE*
Department of ECE, Indian Institute of Science, Bangalore 560012, India

Abstract

In this paper, we analyze the performance of maximum-likelihood (ML) multiuser detection in space-time coded CDMA systems with imperfect channel estimation. A K-user synchronous CDMA system which employs orthogonal space-time block codes with M transmit antennas and N receive antennas is considered. A least-squares estimate of the channel matrix is obtained by sending a sequence of pilot bits from each user. The channel matrix is perturbed by an error matrix which depends on the thermal noise as well as the correlation between the signature waveforms of different users. Because of the linearity of the channel estimation technique, we use the characteristic function of the decision variable to obtain an exact expression for the pairwise error probability (PEP), using which we obtain an upper bound on the bit error rate (BER). The analytical BER bounds are compared with the BER obtained through simulations. The BER bounds are shown to be increasingly tight for large SNR values. It is shown that the degradation in BER performance due to imperfect channel estimation can be compensated by using more number of transmit/receive antennas.

Keywords – *ML Multiuser Detection, space-time codes, CDMA, imperfect channel estimation.*

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1 Introduction

Space-time coded transmission using multiple transmit antennas can offer the benefits of transmit diversity and high data rate transmission on fading channels [1]. Space-time coding applied to code division multiple access (CDMA) systems has been of interest [2]. Multiuser detection schemes, which can significantly enhance the receiver performance and increase the capacity of CDMA systems, have been extensively studied in the literature, mainly for single transmit antenna systems [3]. Multiuser detection schemes and their performances in space-time coded CDMA systems with multiple transmit antennas has been a topic of recent investigations [4],[5],[6],[7]. The performance of the systems considered in [4]-[6] were evaluated mainly through simulations. In [7], Uysal and Georghiades derived an exact analytical expression for the pairwise error probability (PEP) and obtained an approximate bit error rate (BER) expression for a space-time coded CDMA system, but only for a conventional matched filter (MF) detector. Our main focus in this paper is to obtain the PEP and BER expressions for multiuser detection in space-time coded CDMA, particularly when the channel estimates at the receiver are imperfect.

In [8], Taricco and Biglieri obtained an analytical expression for the PEP of space-time codes in a single user system, assuming perfect channel estimation at the receiver. Using this PEP, they obtained bounds on the probability of bit error for maximum-likelihood (ML) detection. In [9], Garg *et al* extended the work in [8] by incorporating imperfect channel estimation in the system model, again for the single user system. For a multiuser system, bounds on the bit error probability of the ML multiuser detection have been derived in [3] (Ch. 4.3) for a 1-Tx/1-Rx antenna system. In this paper, we consider the performance analysis of ML multiuser detection in space-time coded CDMA with multiple transmit and receive antennas. Specifically, we derive an upper bound on the bit error probability of ML multiuser detection for a space-time coded CDMA system for the case of both perfect as well as imperfect channel estimation at the receiver [10],[11]. We consider two channel estimation schemes which require transmission of pilot symbols from different users for the purpose of channel estimation at the receiver. In both schemes, we use the least-squares method for estimation [16]. We consider the least-squares method mainly because of its simplicity and analytical tractability. In the second scheme, we exploit the structure of the channel matrix in such a way that it is computationally less complex than the first scheme. The channel matrix is perturbed by an error matrix which depends on the thermal noise as well as the correlation between the signature waveforms of different users.

Using a discrete-time vector model of the received signal in a space-time coded CDMA system with M transmit and N receive antennas [12], and the characteristic function of the decision variable, we derive an exact expression, in closed-form, for the PEP of the joint data vector of bits from different users. Using this exact PEP expression, we then obtain an upper bound on the average BER. We compare the analytical BER bounds with the BER obtained through simulations, and show that the BER bounds are increasingly tight for large SNR values. It is shown that the degradation in BER performance due to imperfect channel estimation can be compensated by using more number of transmit/receive antennas.

The rest of the paper is organized as follows. In Section 2, we present the system model. In Sections 3 and 4 we present the performance analysis for the channel estimation schemes I and II, respectively. Section 5 presents the performance results and discussions. Conclusions are given in Section 6.

2 System Model

We consider a K-user synchronous CDMA system with M transmit antennas per user. Users transmit data blocks with Q bits per data block. Let b_{iq}, $i \in \{1, 2, ..., K\}$, $q \in \{1, 2, ..., Q\}$, be the q^{th} bit of the i^{th} user, transmitted in a time interval of length T. The bits in a data block are mapped on to the M transmit antennas using real orthogonal space-time block codes (STBC). We assume that the channel fading is quasi-static and the quasi-static interval is

QT time units, where $Q = 2^r$, r being the smallest integer satisfying $Q \geq M$ [2]. For square real orthogonal STBC with $M = Q = 8$, the transmission matrix $\mathbf{X}$ is given by [13]

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ x_2 & -x_1 & -x_4 & x_3 & -x_6 & x_5 & x_8 & -x_7 \\ x_3 & x_4 & -x_1 & -x_2 & -x_7 & -x_8 & x_5 & x_6 \\ x_4 & -x_3 & x_2 & -x_1 & -x_8 & x_7 & -x_6 & x_5 \\ x_5 & x_6 & x_7 & x_8 & -x_1 & -x_2 & -x_3 & -x_4 \\ x_6 & -x_5 & x_8 & -x_7 & x_2 & -x_1 & x_4 & -x_3 \\ x_7 & -x_8 & -x_5 & x_6 & x_3 & -x_4 & -x_1 & x_2 \\ x_8 & x_7 & -x_6 & -x_5 & x_4 & x_3 & -x_2 & -x_1 \end{bmatrix}. \tag{1}$$

In the above transmission matrix, the columns represent the transmit antenna index and the rows represent the bit interval index. For BPSK modulation, which is assumed in this paper, $x_i \in \{1, -1\}$. The transmission matrix $\mathbf{X}$ for other real orthogonal designs for $M, Q < 8$ can be obtained as the upper leftmost submatrix of $\mathbf{X}$ of order $Q \times M$. In the following, we illustrate the received signal model for $M = Q = 2$ (extension of the model for other values of $M, Q \leq 8$ is straightforward [12]). For $M = Q = 2$, the transmission matrix is given by

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}. \tag{2}$$

Each user's data is spread by its assigned spreading (signature) waveform before transmission. The received signal on a receive antenna can be written using (2) as

$$y(t) = y_1(t) + y_2(t) + z(t), \tag{3}$$

$$y_1(t) = \sum_{i=1}^{K} A_{i1} h_{i1} \left\{ b_{i1} s_{i1} + b_{i2} s_{i2} \right\}, \tag{4}$$

$$y_2(t) = \sum_{i=1}^{K} A_{i2} h_{i2} \left\{ b_{i2} s_{i1} - b_{i1} s_{i2} \right\}. \tag{5}$$

In the above, $y_p(t)$, $p \in \{1, 2\}$ is the received signal due to the p^{th} transmit antenna, A_{ip} is the transmit amplitude on the p^{th} transmit antenna of the i^{th} user, h_{ip} is the complex channel gain from the p^{th} transmit antenna of the i^{th} user, and s_{iq} represents the signature waveform of the i^{th} user for the q^{th} bit in a data block, $q \in \{1, 2\}$, given by $s_{iq} = s_i(t - \overline{q-1}T)$, where $s_i(t)$ is a unit energy signature waveform of the ith user, time limited in the interval $[0, T]$ and represented by $s_i(t) = \sum_{l=1}^{N_c} c_{i,l} P_{T_c}(t - lT_c)$, where N_c is the number of chips per bit interval, T_c is one chip interval (i.e., $N_c = T/T_c$), $c_{i,l} \in \{+1, -1\}$ denotes the lth bit of the ith user, $P_{T_c}(t)$ denotes the chip waveform given by $P_{T_c}(t) = 1$ for $0 \leq t < T_c$ and 0 otherwise. Also, $z(t)$ is a zero mean complex Gaussian noise process with variance $2\sigma^2$.

The demodulator on each receive antenna uses a bank of K matched filters, each matched to a different user's signature waveform. The received signal at the output of the matched filters can be written as

$$y_{jq} = \int_0^{QT} y(t) s_{jq}(t) dt. \tag{6}$$

The corresponding noise signal is given by

$$\eta_{jq} = \int_0^{QT} z(t) s_{jq}(t) dt, \tag{7}$$

where j=$1, 2, ..., K$, and $q \in \{1, 2\}$. We define matrix $\mathbf{R}$ as

$$\mathbf{R} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1K} \\ \rho_{12} & 1 & \cdots & \rho_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{1K} & \rho_{2K} & \cdots & 1 \end{bmatrix}, \tag{8}$$

where $\rho_{jk} = \int_0^T s_j(t) s_k(t) dt$. Here, we assume that the signature waveforms are linearly independent so that $\mathbf{R}$ is positive definite. Now, define the matrix $\mathbf{H}$ (of order $QK \times QK$), for $M = Q = 8$, as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_4 & \mathbf{H}_5 & \mathbf{H}_6 & \mathbf{H}_7 & \mathbf{H}_8 \\ -\mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_4 & -\mathbf{H}_3 & \mathbf{H}_6 & -\mathbf{H}_5 & -\mathbf{H}_8 & \mathbf{H}_7 \\ -\mathbf{H}_3 & -\mathbf{H}_4 & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_7 & \mathbf{H}_8 & -\mathbf{H}_5 & -\mathbf{H}_6 \\ -\mathbf{H}_4 & \mathbf{H}_3 & -\mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_8 & -\mathbf{H}_7 & \mathbf{H}_6 & -\mathbf{H}_5 \\ -\mathbf{H}_5 & -\mathbf{H}_6 & -\mathbf{H}_7 & -\mathbf{H}_8 & \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \mathbf{H}_4 \\ -\mathbf{H}_6 & \mathbf{H}_5 & -\mathbf{H}_8 & \mathbf{H}_7 & -\mathbf{H}_2 & \mathbf{H}_1 & -\mathbf{H}_4 & \mathbf{H}_3 \\ -\mathbf{H}_7 & \mathbf{H}_8 & \mathbf{H}_5 & -\mathbf{H}_6 & -\mathbf{H}_3 & \mathbf{H}_4 & \mathbf{H}_1 & -\mathbf{H}_2 \\ -\mathbf{H}_8 & -\mathbf{H}_7 & \mathbf{H}_6 & \mathbf{H}_5 & -\mathbf{H}_4 & -\mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 \end{bmatrix}, \tag{9}$$

where $\mathbf{H}_q = diag[h_{1q}, \ldots, h_{Kq}]$. Also, define the matrix $\mathbf{C}$ of size $QK \times QK$ as

$$\mathbf{C} = \begin{bmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots \\ \mathbf{0} & \mathbf{R} & \mathbf{0} & \mathbf{0} & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{0} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{R} \end{bmatrix}. \tag{10}$$

For values of M and Q other than 8 (i.e., $M, Q < 8$), $\mathbf{H}$ is given by the upper leftmost submatrix of order $QK \times QK$ in (9). For the case of $M \notin \{1, 2, 4, 8\}$, $M < Q$. Therefore, only the elements $\mathbf{H}_q$, $q = 1, 2, ..., M$, are non-zero, i.e., $\mathbf{H}_q = \mathbf{0}$ for $M < q \leq Q$. The non-zero entries of the channel matrix $\mathbf{H}$ are assumed to be i.i.d, zero-mean complex circular Gaussian random variables (i.e., Rayleigh fading) with variance Ω. With the following definitions,

$$\begin{aligned}
\mathbf{y}_q &= [y_{1q}, \ldots, y_{Kq}]^T, & (11)\\
\mathbf{y} &= [\mathbf{y}_1^T, \mathbf{y}_2^T, \ldots, \mathbf{y}_Q^T]^T, & (12)\\
\mathbf{b}_q &= [A_{1q}b_{1q}, \ldots, A_{Kq}b_{Kq}]^T, & (13)\\
\mathbf{b} &= [\mathbf{b}_1^T, \ldots, \mathbf{b}_Q^T]^T, & (14)\\
\boldsymbol{\eta}_q &= [\eta_{1q}, \ldots, \eta_{Kq}]^T, & (15)\\
\boldsymbol{\eta} &= [\boldsymbol{\eta}_1^T, \ldots, \boldsymbol{\eta}_Q^T]^T, & (16)
\end{aligned}$$

we can write

$$\mathbf{y} = \mathbf{CHb} + \boldsymbol{\eta}, \tag{17}$$

which is the generalized vector model for the received signal at the output of the matched filter when real orthogonal space-time block codes are used at the transmitter.

Since $\mathbf{R}$ is positive definite, the correlation matrix $\mathbf{C}$ is also positive definite. Doing the Cholesky decomposition of $\mathbf{C}$

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}, \tag{18}$$

we can write an alternate form of $\mathbf{y}$ as

$$\hat{\mathbf{y}} = (\mathbf{F}^T)^{-1} \mathbf{y} \ = \mathbf{FHb} + \mathbf{n}, \tag{19}$$

where $E[\mathbf{n}] = \mathbf{0}_{QK\times 1}$, $E[\mathbf{n}\mathbf{n}^\dagger] = 2\sigma^2 \mathbf{I}_{QK}$, where $(.)^\dagger$ represents the Hermitian operation and $\mathbf{I}$ is the identity matrix. We will use the vector $\hat{\mathbf{y}}$ in all the analyses in the subsequent sections.

3 Channel Estimation - I

In the channel estimation scheme I, each user is assumed to transmit a sequence of Q pilot bits L_p times for the purpose of channel estimation at the receiver. From (19), the received vector due to the k^{th} set of Q pilot bits per user is obtained as

$$\hat{\mathbf{y}}_k = \mathbf{FHb}_k + \mathbf{n}_k, 1 \leq k \leq L_p. \tag{20}$$

Let the matrix $\mathbf{B}_p$ of dimension $QK \times L_p$ denote the sequence of composite pilot vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_{L_p}$. $\mathbf{B}_p$ is given by

$$\mathbf{B}_p = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_{L_p} \end{bmatrix}, \tag{21}$$

and $\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_{L_p}$ are complex Gaussian random vectors such that

$$E\left[\mathbf{n}_p\right] = \mathbf{0}_{QK\times 1}, E\left[\mathbf{n}_p \mathbf{n}_p^\dagger\right] = 2\sigma^2 \mathbf{I}_{QK}. \tag{22}$$

The received pilot matrix $\widehat{\mathbf{Y}}_p$ can then be written as

$$\widehat{\mathbf{Y}}_p = \mathbf{F}\mathbf{H}\mathbf{B}_p + \mathbf{N}_p, \tag{23}$$

where $\mathbf{N}_p = \left[\begin{array}{cccc} \mathbf{n}_1 & \mathbf{n}_2 & \cdots & \mathbf{n}_{L_p} \end{array} \right]$. The least-squares estimate of the channel matrix $\mathbf{H}$ can be obtained as [16]

$$\widehat{\mathbf{H}} = \mathbf{F}^{-1}\widehat{\mathbf{Y}}_p\mathbf{B}_p^T \left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}. \tag{24}$$

For the above equation to hold, the matrix $\left(\mathbf{B}_p\mathbf{B}_p^T\right)$ has to be invertible, i.e., $L_p \geq QK$. From (23) and (24),

$$\widehat{\mathbf{H}} = \mathbf{H} + \mathbf{F}^{-1}\mathbf{N}_p\mathbf{B}_p^T \left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}, \tag{25}$$

which gives an estimate of the channel matrix $\mathbf{H}$. We will use this estimated channel matrix $\widehat{\mathbf{H}}$ (i.e., imperfect channel estimates) in the following BER analysis for ML multiuser detection.

3.1 ML Criterion

Using the vector representation of the multiuser received signal in (19), the ML multiuser detection criterion can be written as follows. From (24), we obtain the estimates of the channel gains at the receiver. The ML estimate of the transmitted bit vector, $\mathbf{b}$, (comprising the bits from all users) is then given by

$$\tilde{\mathbf{b}} = arg\left\{\min_{\mathbf{w}} \sum_{j=1}^{N} \|\hat{\mathbf{y}}^{(j)} - \mathbf{F}\widehat{\mathbf{H}}^{(j)}\mathbf{w}\|^2\right\}, \tag{26}$$

where the superscript (j) in $\mathbf{y}$ and $\widehat{\mathbf{H}}$ denotes the receive antenna index, $\|\cdot\|$ operator denotes Euclidean vector norm, i.e., $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$, and the $\min_{\mathbf{w}}$ is over all possible bit vectors of length QK. Substituting (19) in (26),

$$\tilde{\mathbf{b}} = arg\left\{\min_{\mathbf{w}} \sum_{j=1}^{N} \|\mathbf{F}\mathbf{H}^{(j)}(\mathbf{b}-\mathbf{w}) + \mathbf{n}^{(j)} - \mathbf{N}_p^{(j)}\mathbf{B}_p^T \left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}\mathbf{w}\|^2\right\}. \tag{27}$$

Note that when the channel estimates are perfect, the ML criterion in (27) becomes

$$\tilde{\mathbf{b}} = arg\left\{\min_{\mathbf{w}} \sum_{j=1}^{N} \|\mathbf{F}\mathbf{H}^{(j)}(\mathbf{b}-\mathbf{w}) + \mathbf{n}^{(j)}\|^2\right\}. \tag{28}$$

3.2 BER Analysis

In this section, we analyze the bit error performance of the ML multiuser detection scheme in (27). We first derive an expression for the PEP, $P(\mathbf{b} \rightarrow \tilde{\mathbf{b}})$, the probability that the transmitted bit vector $\mathbf{b}$ is wrongly decoded as $\tilde{\mathbf{b}}$, and then obtain a bound on the bit error probability. The PEP is given by

$$P(\mathbf{b} \rightarrow \tilde{\mathbf{b}}) = \Pr\left\{\sum_{j=1}^{N} \|\mathbf{F}\mathbf{H}^{(j)}(\mathbf{b}-\tilde{\mathbf{b}}) + \mathbf{n}^{(j)} - \mathbf{N}_p^{(j)}\mathbf{B}_p^T\left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}\tilde{\mathbf{b}}\|^2 - \|\mathbf{n}^{(j)} - \mathbf{N}_p^{(j)}\mathbf{B}_p^T\left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}\mathbf{b}\|^2 < 0\right\}. \tag{29}$$

Define the metric D as

$$D = \sum_{j=1}^{N} \|\mathbf{u}^{(j)}\|^2 - \|\mathbf{v}^{(j)}\|^2, \tag{30}$$

where

$$\begin{aligned}
\mathbf{u}^{(j)} &= \mathbf{F}\mathbf{H}^{(j)}(\mathbf{b}-\tilde{\mathbf{b}}) + \mathbf{n}^{(j)} - \mathbf{N}_p^{(j)}\mathbf{B}_p^T\left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}\tilde{\mathbf{b}} \\
&= \mathbf{F}\mathbf{H}^{(j)}(\mathbf{b}-\tilde{\mathbf{b}}) + \mathbf{n}^{(j)} - \mathbf{N}_p^{(j)}\tilde{\mathbf{c}}, \\
\mathbf{v}^{(j)} &= \mathbf{n}^{(j)} - \mathbf{N}_p^{(j)}\mathbf{B}_p^T\left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}\mathbf{b} \\
&= \mathbf{n}^{(j)} - \mathbf{N}_p^{(j)}\mathbf{c}, \\
\tilde{\mathbf{c}} &= \mathbf{B}_p^T\left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}\tilde{\mathbf{b}}, \\
\mathbf{c} &= \mathbf{B}_p^T\left(\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}\mathbf{b}.
\end{aligned} \tag{31}$$

Note that, for the case of perfect channel estimates,

$$\mathbf{c} = \tilde{\mathbf{c}} = \mathbf{0}. \tag{32}$$

Now, (30) can be written in the form

$$D = \mathbf{V}^{\dagger}\mathbf{S}\mathbf{V}, \tag{33}$$

where

$$\mathbf{V} = \begin{pmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(N)} \\ \mathbf{v}^{(1)} \\ \vdots \\ \mathbf{v}^{(N)} \end{pmatrix}, \tag{34}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_{QKN} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{QKN}. \end{bmatrix}. \tag{35}$$

The decision variable D in (33) is in Hermitian quadratic form in the complex Gaussian random vector $\mathbf{V}$. This form, from a result in [14], allows us to write the characteristic function of D, $\Phi_D(j\omega)$, in closed-form.

The characteristic function and the subsequent derivation of the PEP in closed-form is given in Appendix A. The closed-form expression for the PEP for the case of imperfect channel estimation scheme I can be obtained as (see Appendix A)

$$P(\mathbf{b} \to \tilde{\mathbf{b}}) = \sum_j \frac{(-\lambda_j)^{MN(2K-g_j)}}{\prod_i (\rho_i - \lambda_j)^{MNr_i} \prod_{k \neq j} (\lambda_k - \lambda_j)^{MNg_k}}$$
$$\cdot \sum_{\substack{(l_1, \cdots, l_{MNg_j-1}) \\ 0 \leq l_1, \cdots, l_{MNg_j-1} \leq MNg_j - 1 \\ l_1 + 2l_2 + \cdots + (MNg_j - 1) l_{MNg_j-1} = MNg_j - 1}} \prod_{m=1}^{MNg_j-1} \frac{1}{l_m!} \cdot \left[\frac{1}{m} + \frac{MN}{m} \left(\sum_i \frac{r_i \rho_i^m}{(\rho_i - \lambda_j)^m} + \sum_{k \neq j} \frac{g_k \lambda_k^m}{(\lambda_k - \lambda_j)^m} \right) \right]^{l_m}, \tag{36}$$

where K is the number of users, M is the number of transmit antennas per user, N is the number of antennas at the receiver, and other variables are as defined in Appendix A. Note that (36) can be used for the computation of the PEP when the channel estimates are perfect, by substituting $\beta = \kappa = \epsilon = 1$ in (70), which is obvious from (28) and (32).

From the PEP expression in (36), we obtain an upper bound on the average BER. The derivation of the upper bound on the BER is given in Appendix B. The expression for the bound on the BER is given by (see Appendix B)

$$P(e_{iq}) \leq \frac{1}{2^{QK}} \Bigg[\sum_{j=1}^{2^{QK-1}} \sum_{k=1}^{2^{QK-1}} P(\mathbf{b}^{(j)} \rightarrow \mathbf{b}^{(k)} | b_{iq}^{(j)} = 1, b_{iq}^{(k)} = -1) + \sum_{k=1}^{2^{QK-1}} \sum_{j=1}^{2^{QK-1}} P(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(j)} | b_{iq}^{(k)} = 1, b_{iq}^{(j)} = -1) \Bigg]. \quad (37)$$

Because of symmetry, for the case of perfect channel estimation, the expression for the bound on the BER in (37) can be simplified to

$$P(e_{iq}) \leq \frac{1}{2^{QK-1}} \left[\sum_{j=1}^{2^{QK-1}} \sum_{k=1}^{2^{QK-1}} P(\mathbf{b}^{(j)} \rightarrow \mathbf{b}^{(k)} | b_{iq}^{(j)} = 1, b_{iq}^{(k)} = -1) \right]. \quad (38)$$

4 Channel Estimation - II

The channel estimation scheme I analyzed in the previous section suffers from two disadvantages. Firstly, it requires $QK \times L_p$ pilot bits for estimation. Secondly, the pseudo-inverse of a $QK \times L_p$ matrix, where $L_p \geq QK$, has to exist, which is a difficult proposition. We address these two issues through channel estimation scheme II. In channel estimation scheme II, each user is assumed to transmit a sequence of Q pilot bits only once for the purpose of channel estimation at the receiver, i.e., a total of QK bits are transmitted. But here we exploit the structure of the channel matrix in such a way that it is computationally less complex than the channel estimation scheme I, as follows. From (19), the received vector due to the Q pilot bits is obtained as

$$\hat{\mathbf{y}}_p = \mathbf{F}\mathbf{H}\mathbf{b}_p + \mathbf{n}_p, \quad (39)$$

where $\mathbf{b}_p$ is the composite pilot vector consisting of the Q pilot bits from each of the K users. Using (58), (39) can be rewritten as

$$\widehat{\mathbf{y}}_p = \mathbf{F}\mathbf{B}_p\mathbf{h} + \mathbf{n}_p, \tag{40}$$

where $\mathbf{B}_p$ has properties similar to (59) and $\mathbf{h}$ is a complex Gaussian random vector such that $E[\mathbf{h}] = \mathbf{0}_{QK\times 1}$ and $E[\mathbf{h}\mathbf{h}^\dagger] = \Omega\mathbf{I}_{QK}$, and

$$E\left[\mathbf{n}_p\right] = \mathbf{0}_{QK\times 1}, E\left[\mathbf{n}_p\mathbf{n}_p^\dagger\right] = 2\sigma^2\mathbf{I}_{QK}. \tag{41}$$

The least-squares estimate of the channel vector $\mathbf{h}$ can then be obtained as

$$\widehat{\mathbf{h}} = \left(\mathbf{F}\mathbf{B}_p\right)^{-1}\widehat{\mathbf{y}}_p. \tag{42}$$

For the above equation to hold, the matrix $\mathbf{B}_p$ has to be invertible. From (40) and (42),

$$\widehat{\mathbf{h}} = \mathbf{h} + \left(\mathbf{F}\mathbf{B}_p\right)^{-1}\mathbf{n}_p. \tag{43}$$

In channel estimation scheme II, the choice of the pilot matrix $\mathbf{B}_p$ is easier than in scheme I because we now have to find an invertible square matrix of size QK, instead of a rectangular matrix of size $QK \times L_p, L_p \geq QK$ which has a pseudo-inverse. In the following subsections, we present the ML criterion and BER analysis for the channel estimation scheme II.

4.1 ML Criterion

Using the vector representation of the multiuser received signal in (19), the ML multiuser detection criterion with channel estimation scheme II can be written as follows. From (42), we obtain the estimates of the channel gains at the receiver. The ML estimate of the transmitted bit vector, $\mathbf{b}$, (comprising the bits from all users) is then given by

$$\begin{aligned}\tilde{\mathbf{b}} &= arg\left\{\min_{\mathbf{w}}\sum_{j=1}^{N}\|\widehat{\mathbf{y}}^{(j)}-\mathbf{F}\widehat{\mathbf{H}}^{(j)}\mathbf{w}\|^2\right\}\\ &= arg\left\{\min_{\mathbf{w}}\sum_{j=1}^{N}\|\widehat{\mathbf{y}}^{(j)}-\mathbf{F}\mathbf{W}\widehat{\mathbf{h}}^{(j)}\|^2\right\},\end{aligned} \quad (44)$$

where the superscript (j) in $\mathbf{y}$ and $\widehat{\mathbf{h}}$ denotes the receive antenna index, and the $\min_{\mathbf{w}}$ is over all possible bit vectors of length QK. The one to one correspondence between vectors $\mathbf{w}, \mathbf{b}$ and $\tilde{\mathbf{b}}$, and matrices $\mathbf{W}, \mathbf{B}$ and $\tilde{\mathbf{B}}$, respectively, is illustrated by (60). Substituting (40) in (44),

$$\tilde{\mathbf{b}} = arg\left\{\min_{\mathbf{w}}\sum_{j=1}^{N}\|\mathbf{F}(\mathbf{B}-\mathbf{W})\mathbf{h}^{(j)}+\mathbf{n}^{(j)}-\mathbf{F}\mathbf{W}\left(\mathbf{F}\mathbf{B}_{\mathbf{p}}\right)^{-1}\mathbf{n}_p^{(j)}\|^2\right\}. \quad (45)$$

4.2 BER Analysis

The PEP, $P(\mathbf{b}\rightarrow\tilde{\mathbf{b}})$ is given by

$$P(\mathbf{b}\rightarrow\tilde{\mathbf{b}}) = \Pr\left\{\sum_{j=1}^{N}\|\mathbf{F}(\mathbf{B}-\tilde{\mathbf{B}})\mathbf{h}^{(j)}+\mathbf{n}^{(j)}-\mathbf{F}\tilde{\mathbf{B}}\left(\mathbf{F}\mathbf{B}_p\right)^{-1}\mathbf{n}_p^{(j)}\|^2-\|\mathbf{n}^{(j)}-\mathbf{F}\mathbf{B}\left(\mathbf{F}\mathbf{B}_p\right)^{-1}\mathbf{n}_p^{(j)}\|^2<0\right\}. \quad (46)$$

Define the metric D as

$$D = \sum_{j=1}^{N}\|\mathbf{u}^{(j)}\|^2-\|\mathbf{v}^{(j)}\|^2, \quad (47)$$

where

$$\begin{aligned}
\mathbf{u}^{(j)} &= \mathbf{F}(\mathbf{B}-\tilde{\mathbf{B}})\mathbf{h}^{(j)} + \mathbf{n}^{(j)} - \mathbf{F}\tilde{\mathbf{B}}\left(\mathbf{F}\mathbf{B}_p\right)^{-1}\mathbf{n}_p^{(j)} \\
&= \mathbf{F}(\mathbf{B}-\tilde{\mathbf{B}})\mathbf{h}^{(j)} + \mathbf{n}^{(j)} - \tilde{\mathbf{A}}\mathbf{n}_p^{(j)}, \\
\mathbf{v}^{(j)} &= \mathbf{n}^{(j)} - \mathbf{F}\mathbf{B}\left(\mathbf{F}\mathbf{B}_p\right)^{-1}\mathbf{n}_p^{(j)} \\
&= \mathbf{n}^{(j)} - \mathbf{A}\mathbf{n}_p^{(j)}, \\
\tilde{\mathbf{A}} &= \mathbf{F}\tilde{\mathbf{B}}\left(\mathbf{F}\mathbf{B}_p\right)^{-1}, \\
\mathbf{A} &= \mathbf{F}\mathbf{B}\left(\mathbf{F}\mathbf{B}_p\right)^{-1}.
\end{aligned} \tag{48}$$

Again, (47) can be written in the form

$$D = \mathbf{V}^{\dagger}\mathbf{S}\mathbf{V}, \tag{49}$$

where

$$\mathbf{V} = \begin{pmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(N)} \\ \mathbf{v}^{(1)} \\ \vdots \\ \mathbf{v}^{(N)} \end{pmatrix}, \tag{50}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_{QKN} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{QKN}. \end{bmatrix}. \tag{51}$$

The decision variable D in (49) is in Hermitian quadratic form in the complex Gaussian random vector $\mathbf{V}$. We again use [14] to write $\Phi_D(j\omega)$, in closed-form. The derivation of the PEP using the characteristic function is given in Appendix C. The closed-form expression for the PEP for the channel estimation scheme II is obtained as (see Appendix C)

$$P(\mathbf{b}\to\tilde{\mathbf{b}}) = \sum_j \frac{(-\lambda_j)^{N(2K-g_j)}}{\prod_i(\rho_i-\lambda_j)^{Nr_i}\prod_{k\neq j}(\lambda_k-\lambda_j)^{Ng_k}}$$

$$\cdot \sum_{\substack{(l_1,\cdots,l_{Ng_j-1}) \\ 0\le l_1,\cdots,l_{Ng_j-1}\le Ng_j-1 \\ l_1+2l_2+\cdots+(Ng_j-1)l_{Ng_j-1}=Ng_j-1}} \prod_{m=1}^{Ng_j-1}\frac{1}{l_m!}\cdot\left[\frac{1}{m}+\frac{N}{m}\left(\sum_i\frac{r_i\rho_i^m}{(\rho_i-\lambda_j)^m}+\sum_{k\neq j}\frac{g_k\lambda_k^m}{(\lambda_k-\lambda_j)^m}\right)\right]^{l_m}, \tag{52}$$

where K is the number of users, M is the number of transmit antennas per user, N is the number of antennas at the receiver, and other variables are as defined in Appendix C.

Using the expression for PEP in the above, we obtain an upper bound on the bit error probability as follows. Let $\mathbf{b}^{(j)}, 1 \leq j \leq 2^{QK}$ be the set of QK-bit vectors comprising of Q bits from each of the K users. Suppose $\mathbf{b}^{(k)}$ was the transmitted vector. Define

$$D_m = \sum_{j=1}^{N} \|\hat{\mathbf{y}}^{(j)} - \mathbf{F}\mathbf{B}^{(m)}\mathbf{h}^{(j)}\|^2, \quad m = 1, 2, \cdots, 2^{QK}, \tag{53}$$

where $\hat{\mathbf{y}}$, $\mathbf{F}$ and $\mathbf{h}$ and $\mathbf{B}^{(m)}$ are as defined in (44). If $\mathbf{b}^{(l)}$ is the received vector, define

$$P_{exact}\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(l)}\right) = \Pr\left(\bigcap_{\substack{m=1\\ m\neq l}}^{2^{QK}} (D_l < D_m)\right). \tag{54}$$

It is noted that the PEP in (52) is nothing but $P\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(l)}\right) = \Pr\left(D_l < D_k\right)$. It is clear that

$$P_{exact}\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(l)}\right) \leq P\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(l)}\right). \tag{55}$$

Let $P(e_{iq})$ denote the probability of error for the q^{th} bit of the i^{th} user, $q = 1, 2, \cdots, Q$ and $i = 1, 2, \cdots, K$. An upper bound on $P(e_{iq})$ is then given by

$$\begin{aligned} P(e_{iq}) \quad \leq \quad & \frac{1}{2^{QK}}\left[\sum_{j=1}^{2^{QK-1}} \sum_{k=1}^{2^{QK-1}} P(\mathbf{b}^{(j)} \rightarrow \mathbf{b}^{(k)} | b_{iq}^{(j)} = 1, b_{iq}^{(k)} = -1)\right. \\ & \left. + \sum_{k=1}^{2^{QK-1}} \sum_{j=1}^{2^{QK-1}} P(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(j)} | b_{iq}^{(k)} = 1, b_{iq}^{(j)} = -1)\right]. \end{aligned} \tag{56}$$

5 Numerical Results

In this section, we present the numerical results of the error performance of the space-time coded ML multiuser detection with perfect as well as imperfect channel estimates. First, we present the error performance of the channel estimation scheme I in Figs. 1-5. Fig. 1 shows both the analytically computed PEP (Eqn. (36)) as well as the PEP obtained through simulations, as a function of SNR for the channel estimation scheme I, for $M = 2$, $N = 1$, $K = 2$, $\rho = 0.2$, transmitted bit vector $\mathbf{b} = (1, 1, 1, -1)^T$, which is erroneously decoded as vector $\tilde{\mathbf{b}} = (-1, -1, 1, -1)^T$. All users are assumed to be received with equal power. As evident from the figure, the analytical and simulation results tally very well, which validates the correctness of the PEP expression in (36) for the channel estimation scheme I. Fig. 2 shows the PEP plots for the cases of both perfect channel estimation as well as imperfect channel estimation scheme I, for $K = 2$, $M = 2$, $N = 1$, $\rho = 0.2$, $\mathbf{b} = (1, 1, 1, 1)^T$ and $\tilde{\mathbf{b}} = (1, 1, -1, -1)^T$. It can be seen that, as expected, the PEP degrades with imperfect channel estimation compared to the perfect channel estimates case. From Fig. 2, it is seen that a PEP of 10^{-2} is achieved at an average SNR of approximately 8 dB, whereas to achieve the same PEP, an average SNR of approximately 12 dB is needed when the channel estimates are imperfect.

Fig. 3 presents the bit error rate performance obtained through the analytical bound (computed using (37)) as well as simulations, for $K = 2$, $M = 2$, $N = 1$ and $\rho = 0.2$. Plots for both perfect as well as imperfect channel estimates I are shown. It can be observed that the analytical BER bounds become increasingly tight for large SNR values ($>$ 10 dB). Also, imperfect channel estimates are seen to degrade the BER performance, as expected. For example, at a BER of 10^{-2}, the performance loss is about 4 dB in the case of imperfect channel estimation I, compared to the perfect channel estimation case.

Figs. 4 and 5 show the bound on the BER as a function of average SNR for $M = 2$, $N = 1, 2$, (fixed number of transmit antennas and varying number of receive antennas), and $N = 2$, $M = 1, 2$, (fixed number of receive antennas and varying number of transmit antennas), respectively, for the cases of perfect as well as imperfect channel estimates I. From Figs. 4 and 5, it is seen that the degradation in BER performance due to imperfect channel estimates can be compensated by using more number of receive/transmit antennas. For example, in Fig. 5, at about 12 dB average SNR and $M = 1$, the BER worsens from approximately 2×10^{-2} to 5×10^{-2} for imperfect channel estimation scheme I compared to the perfect channel estimates case. This loss in performance can be compensated by using $M = 2$ transmit antennas where, even with imperfect channel estimation, a BER of 2×10^{-2} is achieved at the same 12 dB SNR.

In Figs. 6 and 7, we provide the error performance of imperfect channel estimation scheme II. Fig. 6 shows both the analytically computed PEP (from (52)) as well as the PEP obtained through simulations as a function of SNR for the channel estimation scheme II, for $M = 2, N = 1, K = 2$, $\rho = 0.2$, $\mathbf{b} = (1, 1, -1, 1)^T$ and $\tilde{\mathbf{b}} = (-1, -1, -1, -1)^T$. Here again, the good match between analytical and simulation results validates the correctness of the PEP expression in (52). In Fig. 7, we present the behavior of the bound on the probability of bit error for the case of channel estimation II, obtained from (56), with respect to the BER obtained through simulations, for $M = 2$, $N = 1$, $K = 2$ and $\rho = 0.2$. From Fig. 7, it is observed that the bound is quite loose for low SNR values (< 10 dB), but is increasingly tight for high SNR values (> 10 dB).

Finally, Fig. 8 provides the comparison of the analytical BER bounds for the three cases of interest, namely, a) perfect channel estimates b) channel estimation scheme I, and c) channel estimation scheme II, for $M = 2$, $N = 1, 2$, $K = 2$, and $\rho = 0.2$. These bounds are computed using (38), (37), and (56), respectively. It is observed that the performance of channel estimation schemes I and II are quite similar at high SNR values, where the bounds are shown to be tight. Fig. 8 also indicates that the performance degradation due to channel estimation can be compensated by having more number of receive antennas.

6 Conclusions

We analyzed the performance of the optimum ML multiuser detection for a space-time coded CDMA system with and without channel estimation errors at the receiver. We provided an analysis to quantify the effect of imperfect channel estimation on the BER performance of space-time coded ML multiuser detection. We considered two channel estimation schemes which require transmission of pilot symbols from different users for the purpose of channel estimation at the receiver. We derived an exact expression for the PEP, using the characteristic function approach, and using it, derived an upper bound on the BER. We showed that the performance analysis of space-time coded multiuser detection for the case of perfect channel estimation is a special case of imperfect channel estimation. Through simulations, we showed that the analytical BER bounds are tight for large SNR values for the cases of perfect as well as imperfect channel estimation. It was shown that the degradation in the performance of the space-time coded multiuser detector due to channel estimation errors can be compensated by using more number of transmit/receive antennas.

Appendix A - PEP for Channel Estimation I

In this Appendix, we derive the characteristic function and the PEP for ML multiuser detection with channel estimation scheme I. Let

$$\mathbf{T} = E[\mathbf{V}\mathbf{V}^{\dagger}], \tag{57}$$

where $\mathbf{V}$ is given by (34). To evaluate $\mathbf{T}$ in the above, we write $\mathbf{H}^{(j)}\mathbf{b}$ in an alternate form [2]

$$\mathbf{H}^{(j)}\mathbf{b} = \mathbf{B}\mathbf{h}^{(j)}, \tag{58}$$

where $\mathbf{B}$ is a $QK \times QK$ matrix, which for $M = Q = 8$ is defined as

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \mathbf{B}_5 & \mathbf{B}_6 & \mathbf{B}_7 & \mathbf{B}_8 \\ \mathbf{B}_2 & -\mathbf{B}_1 & -\mathbf{B}_4 & \mathbf{B}_3 & -\mathbf{B}_6 & \mathbf{B}_5 & \mathbf{B}_8 & -\mathbf{B}_7 \\ \mathbf{B}_3 & \mathbf{B}_4 & -\mathbf{B}_1 & -\mathbf{B}_2 & -\mathbf{B}_7 & -\mathbf{B}_8 & \mathbf{B}_5 & \mathbf{B}_6 \\ \mathbf{B}_4 & -\mathbf{B}_3 & \mathbf{B}_2 & -\mathbf{B}_1 & -\mathbf{B}_8 & \mathbf{B}_7 & -\mathbf{B}_6 & \mathbf{B}_5 \\ \mathbf{B}_5 & \mathbf{B}_6 & \mathbf{B}_7 & \mathbf{B}_8 & -\mathbf{B}_1 & -\mathbf{B}_2 & -\mathbf{B}_3 & -\mathbf{B}_4 \\ \mathbf{B}_6 & -\mathbf{B}_5 & \mathbf{B}_8 & -\mathbf{B}_7 & \mathbf{B}_2 & -\mathbf{B}_1 & \mathbf{B}_4 & -\mathbf{B}_3 \\ \mathbf{B}_7 & -\mathbf{B}_8 & -\mathbf{B}_5 & \mathbf{B}_6 & \mathbf{B}_3 & -\mathbf{B}_4 & -\mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_8 & \mathbf{B}_7 & -\mathbf{B}_6 & -\mathbf{B}_5 & \mathbf{B}_4 & \mathbf{B}_3 & -\mathbf{B}_2 & -\mathbf{B}_1 \end{bmatrix}, \quad (59)$$

where $\mathbf{B}_q = \mathbf{A}_q diag\{\mathbf{b}_q\}$, $\mathbf{A}_q = diag\{A_{1q}, A_{2q}, \cdots, A_{Kq}\}$, $q = 1, 2, \cdots, Q$.

Defining $\mathbf{h}_q = [h_{1q}, h_{2q}, \cdots, h_{Kq}]^T$ and $\mathbf{h} = [\mathbf{h}_1^T, \mathbf{h}_2^T, \cdots, \mathbf{h}_Q^T]^T$, $E[\mathbf{h}] = \mathbf{0}_{QK\times 1}$ and $E[\mathbf{h}\mathbf{h}^\dagger] = \Omega\mathbf{I}_{QK}$. For example, when $K = 2, M = Q = 2$, dropping the index j for convenience, we have

$$\begin{aligned} \mathbf{Hb} &= \begin{pmatrix} h_{11} & 0 & h_{12} & 0 \\ 0 & h_{21} & 0 & h_{22} \\ -h_{12} & 0 & h_{11} & 0 \\ 0 & -h_{22} & 0 & h_{21} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} & 0 & b_{12} & 0 \\ 0 & b_{21} & 0 & b_{22} \\ b_{12} & 0 & -b_{11} & 0 \\ 0 & b_{22} & 0 & -b_{21} \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{21} \\ h_{12} \\ h_{22} \end{pmatrix} \\ &= \mathbf{Bh}. \end{aligned} \quad (60)$$

For values of M and Q other than 8 $(M, Q < 8)$, $\mathbf{B}$ is obtained as follows. For $M = Q \in \{1, 2, 4\}$, $\mathbf{B}$ is given by the upper leftmost submatrix of order $QK \times QK$ in (59). For $M \notin \{1, 2, 4, 8\}$, $M < Q$. In this case, $\mathbf{B}$ is given by the $QK \times QK$ upper leftmost submatrix in (59) with all the entries in the q^{th} column $(M < q \leq Q)$ as zeros. Also, let $\beta = \left(1 + \tilde{\mathbf{c}}^T\tilde{\mathbf{c}}\right)$, $\kappa = \left(1 + \tilde{\mathbf{c}}^T\mathbf{c}\right)$ and $\epsilon = \left(1 + \mathbf{c}^T\mathbf{c}\right)$. With the above definitions, we obtain

$$E\left[\mathbf{u}^{(i)}\mathbf{u}^{(j)\dagger}\right] = \begin{cases} \mathbf{0} & i \neq j \\ \Omega\mathbf{F}(\mathbf{B} - \tilde{\mathbf{B}})(\mathbf{B} - \tilde{\mathbf{B}})^T\mathbf{F}^T + 2\sigma^2\beta\mathbf{I}_{QK} & i = j \end{cases} \quad (61)$$

$$E[\mathbf{u}^{(i)}\mathbf{v}^{(j)\dagger}] = \begin{cases} \mathbf{0} & i \neq j \\ 2\sigma^2\kappa\mathbf{I}_{QK} & i = j \end{cases} \quad (62)$$

$$E[\mathbf{v}^{(i)}\mathbf{u}^{(j)\dagger}] = \begin{cases} \mathbf{0} & i \neq j \\ 2\sigma^2\kappa\mathbf{I}_{QK} & i = j \end{cases} \quad (63)$$

$$E[\mathbf{v}^{(i)}\mathbf{v}^{(j)\dagger}] = \begin{cases} \mathbf{0} & i \neq j \\ 2\sigma^2\epsilon\mathbf{I}_{QK} & i = j, \end{cases} \quad (64)$$

from which $\mathbf{T}$ can be evaluated. Now, the characteristic function of D, $\Phi_D(j\omega)$ can be written as (Ref. [14], Eqn. (4.a))

$$\Phi_D(j\omega) = \frac{1}{|\mathbf{I}_{2NQK} - 2j\omega\sigma^2\mathbf{G}|}, \tag{65}$$

where $\mathbf{G} = \mathbf{TS}$, and $\mathbf{S}$ is given by (35). From (61), (62), (63) and (64), we can write $\mathbf{G}$ as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_N \otimes \left(\frac{\Omega}{2\sigma^2}\mathbf{F}(\mathbf{B}-\tilde{\mathbf{B}})(\mathbf{B}-\tilde{\mathbf{B}})^T\mathbf{F}^T + \beta\mathbf{I}_{QK}\right) & -\mathbf{I}_N \otimes \kappa\mathbf{I}_{QK} \\ \mathbf{I}_N \otimes \kappa\mathbf{I}_{QK} & -\mathbf{I}_N \otimes \epsilon\mathbf{I}_{QK} \end{bmatrix}. \tag{66}$$

Defining $\widehat{\mathbf{G}}$ as

$$\widehat{\mathbf{G}} = \begin{bmatrix} (\frac{\Omega}{2\sigma^2}\mathbf{F}(\mathbf{B}-\tilde{\mathbf{B}})(\mathbf{B}-\tilde{\mathbf{B}})^T\mathbf{F}^T + \beta\mathbf{I}_{QK}) & -\kappa\mathbf{I}_{QK} \\ \kappa\mathbf{I}_{QK} & -\epsilon\mathbf{I}_{QK} \end{bmatrix}, \tag{67}$$

(65) can be written as

$$\Phi_D(j\omega) = \frac{1}{|\mathbf{I}_{2QK} - 2j\omega\sigma^2\widehat{\mathbf{G}}|^N} = \prod_{i=1}^{2QK} \frac{1}{|1 - 2j\omega\sigma^2\widehat{\lambda}_i|^N}, \tag{68}$$

where $\widehat{\lambda}_1, \cdots, \widehat{\lambda}_{2QK}$ are the eigenvalues of $\widehat{\mathbf{G}}$. For the case when the amplitudes of all bits from all the users are the same, i.e., $A_{iq} = A_{jq} = A$, $i, j = 1, 2, \cdots, K$, $q = 1, 2, \cdots, Q$, and $M = Q$, (68) can be written in the form

$$\Phi_D(j\omega) = \frac{1}{|\mathbf{I}_{2K} - 2j\omega\sigma^2\tilde{\mathbf{G}}|^{MN}} = \prod_{i=1}^{2K} \frac{1}{|1 - 2j\omega\sigma^2\lambda_i|^{MN}}, \tag{69}$$

where $\tilde{\mathbf{G}}$ is given by

$$\tilde{G} = \begin{bmatrix} \frac{\Omega A^2}{2\sigma^2}\mathbf{P}\mathbf{\Lambda}\mathbf{P}^T + \beta\mathbf{I}_K & -\kappa\mathbf{I}_K \\ \kappa\mathbf{I}_K & -\epsilon\mathbf{I}_K \end{bmatrix}, \tag{70}$$

where $\mathbf{P}$ is the Cholesky decomposition of the $\mathbf{R}$ matrix (i.e., $\mathbf{R} = \mathbf{P}^T\mathbf{P}$), $\mathbf{\Lambda}$ is given by

$$\mathbf{\Lambda} = \frac{1}{A^2}\sum_{i=1}^{Q}(\mathbf{B}_i - \tilde{\mathbf{B}}_i)^2, \tag{71}$$

and $\lambda_1, \cdots, \lambda_{2K}$ are the eigenvalues $\tilde{\mathbf{G}}$. Substituting $z = 2j\omega\sigma^2$, we have

$$\Phi_D(z) = \prod_{i=1}^{2K} \frac{1}{(1 - z\lambda_i)^{MN}}. \tag{72}$$

From the above characteristic function of D, the PEP in (29) can be obtained as [9],[15]

$$P(\mathbf{b} \to \tilde{\mathbf{b}}) = -\sum_k \frac{1}{(p_k - 1)!}\frac{d^{p_k-1}}{dz^{p_k-1}}\left\{(z - \lambda_k)^{p_k}\frac{\Phi_D(z)}{z}\right\}, \tag{73}$$

where λ_k are the negative eigenvalues of $\tilde{G}$, $Re(\lambda_k) < 0$, and p_k is the multiplicity of λ_k. We obtain (73) in closed-form as follows. The characteristic equation of $\tilde{\mathbf{G}}$ is given by

$$det|\lambda \mathbf{I}_{2K} - \tilde{\mathbf{G}}| = det \begin{vmatrix} (\lambda-\beta)\mathbf{I}_K - \gamma \mathbf{J} & \kappa \mathbf{I}_K \\ -\kappa \mathbf{I}_K & (\lambda+\epsilon)\mathbf{I}_K \end{vmatrix} = 0, \tag{74}$$

where $\gamma = \frac{\Omega A^2}{2\sigma^2}$ is the average SNR, and $\mathbf{J} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T$. Eqn. (74) can be shown to reduce to the form [16]

$$det|(\lambda-\beta)(\lambda+\epsilon)\mathbf{I}_K - \gamma(\lambda+\epsilon)\mathbf{J} + \kappa^2 \mathbf{I}_K| = 0. \tag{75}$$

If $\mu_1, \cdots, \mu_L$ are the L distinct eigenvalues of $\mathbf{J}$, each with multiplicity v_i, i.e. $\sum_{i=1}^{L} v_i = 2K$, then (75) reduces to

$$\prod_{i=1}^{L} \left\{ \lambda^2 - (\beta - \epsilon + \gamma\mu_i)\lambda - (\beta\epsilon - \kappa^2 + \gamma\mu_i\epsilon) \right\}^{v_i} = 0. \tag{76}$$

From Sylvester's Law of Inertia [17], the eigenvalues of $\mathbf{J}$ are non-negative (i.e., $\mu_i \geq 0$). Hence, the roots of (76) are all real. Denote the negative roots as λ_j, with multiplicities $g_j, j = 1, 2, \cdots, L_N$ and the non-negative roots as ρ_i, with multiplicities $r_i, i = 1, 2, \cdots, L_P$, so that $\sum_j g_j + \sum_i r_i = 2K$. With this, we can now follow the steps similar to the ones in [9], and obtain the closed-form expression for the PEP as

$$P(\mathbf{b} \rightarrow \tilde{\mathbf{b}}) = \sum_j \frac{(-\lambda_j)^{MN(2K-g_j)}}{\prod_i (\rho_i - \lambda_j)^{MNr_i} \prod_{k \neq j} (\lambda_k - \lambda_j)^{MNg_k}}$$

$$\cdot \sum_{\substack{(l_1, \cdots, l_{MNg_j-1}) \\ 0 \leq l_1, \cdots, l_{MNg_j-1} \leq MNg_j - 1 \\ l_1 + 2l_2 + \cdots + (MNg_j - 1) l_{MNg_j-1} = MNg_j - 1}} \prod_{m=1}^{MNg_j-1} \frac{1}{l_m!} \cdot \left[\frac{1}{m} + \frac{MN}{m} \left(\sum_i \frac{r_i \rho_i^m}{(\rho_i - \lambda_j)^m} + \sum_{k \neq j} \frac{g_k \lambda_k^m}{(\lambda_k - \lambda_j)^m} \right) \right]^{l_m}. \tag{77}$$

Appendix B - Bound on the BER

In this Appendix, we derive an upper bound on the average BER for ML multiuser detection with channel estimation scheme I. Using the expression for PEP in (77), we obtain an upper bound on the bit error probability as follows. Let $\mathbf{b}^{(j)}, 1 \leq j \leq 2^{QK}$ be the set of QK-bit vectors comprising of Q bits from each of the K users. Suppose $\mathbf{b}^{(k)}$ was the transmitted vector. Define

$$D_m = \sum_{j=1}^{N} \|\widehat{\mathbf{y}}^{(j)} - \mathbf{F}\mathbf{H}^{(j)}\mathbf{b}^{(m)}\|^2, \quad m = 1, 2, \cdots, 2^{QK}, \tag{78}$$

where $\widehat{\mathbf{y}}$, $\mathbf{F}$ and $\mathbf{H}$ are as defined in (26). If $\mathbf{b}^{(l)}$ is the received vector, define

$$P_{exact}\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(l)}\right) = \Pr\left(\bigcap_{\substack{m=1\\ m\neq l}}^{2^{QK}} (D_l < D_m)\right). \tag{79}$$

It is noted that the PEP in (77) is nothing but

$$P\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(l)}\right) = \Pr\left(D_l < D_k\right). \tag{80}$$

It is clear that

$$P_{exact}\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(l)}\right) \leq P\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(l)}\right). \tag{81}$$

Let $P(e_{iq})$ denote the probability of error for the q^{th} bit of the i^{th} user, $q = 1, 2, \cdots, Q$ and $i = 1, 2, \cdots, K$. $P(e_{iq})$ is then given by

$$\begin{aligned} P(e_{iq}) \quad = \quad & \sum_{j=1}^{2^{QK-1}} P(e_{iq}|\mathbf{b}^{(j)}, b_{iq}^{(j)} = 1)P(\mathbf{b}^{(j)}, b_{iq}^{(j)} = 1) \\ & + \sum_{k=1}^{2^{QK-1}} P(e_{iq}|\mathbf{b}^{(k)}, b_{iq}^{(k)} = -1)P(\mathbf{b}^{(k)}, b_{iq}^{(k)} = -1). \end{aligned} \tag{82}$$

$P(e_{iq}|\mathbf{b}^{(j)}, b_{iq}^{(j)} = \pm 1)$ and $P(\mathbf{b}^{(j)}, b_{iq}^{(j)} = \pm 1)$ are then given by

$$P\left(e_{iq}|\mathbf{b}^{(j)}, b_{iq}^{(j)} = 1\right) = \sum_{k=1}^{2^{QK-1}} P_{exact}\left(\mathbf{b}^{(j)} \rightarrow \mathbf{b}^{(k)}|b_{iq}^{(j)} = 1, b_{iq}^{(k)} = -1\right), \tag{83}$$

$$P\left(e_{iq}|\mathbf{b}^{(k)}, b_{iq}^{(k)} = -1\right) = \sum_{j=1}^{2^{QK-1}} P_{exact}\left(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(j)}|b_{iq}^{(j)} = 1, b_{iq}^{(k)} = -1\right), \tag{84}$$

$$P(\mathbf{b}^{(j)}, b_{iq}^{(j)} = 1) = P(\mathbf{b}^{(k)}, b_{iq}^{(k)} = -1) = \frac{1}{2^{QK}}. \tag{85}$$

From (81), (82), (83), (84) and(85), an upper bound on the bit error probability $P(e_{iq})$ is obtained as

$$\begin{aligned} P(e_{iq}) \quad \leq \quad & \frac{1}{2^{QK}}\Bigg[\sum_{j=1}^{2^{QK-1}} \sum_{k=1}^{2^{QK-1}} P(\mathbf{b}^{(j)} \rightarrow \mathbf{b}^{(k)}|b_{iq}^{(j)} = 1, b_{iq}^{(k)} = -1) \\ & + \sum_{k=1}^{2^{QK-1}} \sum_{j=1}^{2^{QK-1}} P(\mathbf{b}^{(k)} \rightarrow \mathbf{b}^{(j)}|b_{iq}^{(k)} = 1, b_{iq}^{(j)} = -1)\Bigg]. \end{aligned} \tag{86}$$

Appendix C - PEP for Channel Estimation II

In this Appendix, we derive the characteristic function and PEP for ML multiuser detection with channel estimation scheme II. Let

$$\mathbf{T} = E[\mathbf{V}\mathbf{V}^\dagger], \tag{87}$$

where $\mathbf{V}$ is given by (50). To evaluate $\mathbf{T}$ in the above,

$$E\left[\mathbf{u}^{(i)}\mathbf{u}^{(j)\dagger}\right] = \begin{cases} \mathbf{0} & i \neq j \\ \Omega\mathbf{F}(\mathbf{B} - \tilde{\mathbf{B}})(\mathbf{B} - \tilde{\mathbf{B}})^T\mathbf{F}^T + 2\sigma^2\left(\mathbf{I}_{QK} + \tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\right) & i = j \end{cases} \tag{88}$$

$$E[\mathbf{u}^{(i)}\mathbf{v}^{(j)\dagger}] = \begin{cases} \mathbf{0} & i \neq j \\ 2\sigma^2\left(\mathbf{I}_{QK} + \tilde{\mathbf{A}}\mathbf{A}^T\right) & i = j \end{cases} \tag{89}$$

$$E[\mathbf{v}^{(i)}\mathbf{u}^{(j)\dagger}] = \begin{cases} \mathbf{0} & i \neq j \\ 2\sigma^2\left(\mathbf{I}_{QK} + \mathbf{A}\tilde{\mathbf{A}}^T\right) & i = j \end{cases} \tag{90}$$

$$E[\mathbf{v}^{(i)}\mathbf{v}^{(j)\dagger}] = \begin{cases} \mathbf{0} & i \neq j \\ 2\sigma^2\left(\mathbf{I}_{QK} + \mathbf{A}\mathbf{A}^T\right) & i = j, \end{cases} \tag{91}$$

from which $\mathbf{T}$ can be evaluated. Now, the characteristic function of D, $\Phi_D(j\omega)$ can be written as (Ref. [14], Eqn. (4.a))

$$\Phi_D(j\omega) = \frac{1}{|\mathbf{I}_{2NQK} - 2j\omega\sigma^2\mathbf{G}|}, \tag{92}$$

where $\mathbf{G} = \mathbf{TS}$, and $\mathbf{S}$ is given by (51). From (88), (89), (90), (91), we can write $\mathbf{G}$ as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_N \otimes \left(\frac{\Omega}{2\sigma^2}\mathbf{F}(\mathbf{B}-\tilde{\mathbf{B}})(\mathbf{B}-\tilde{\mathbf{B}})^T\mathbf{F}^T + \mathbf{I}_{QK} + \tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\right) & -\mathbf{I}_N \otimes \left(\mathbf{I}_{QK} + \tilde{\mathbf{A}}\mathbf{A}^T\right) \\ \mathbf{I}_N \otimes \left(\mathbf{I}_{QK} + \mathbf{A}\tilde{\mathbf{A}}^T\right) & -\mathbf{I}_N \otimes \left(\mathbf{I}_{QK} + \mathbf{A}\mathbf{A}^T\right) \end{bmatrix}. \tag{93}$$

Defining $\widehat{\mathbf{G}}$ as

$$\widehat{\mathbf{G}} = \begin{bmatrix} \left(\frac{\Omega}{2\sigma^2}\mathbf{F}(\mathbf{B}-\tilde{\mathbf{B}})(\mathbf{B}-\tilde{\mathbf{B}})^T\mathbf{F}^T + \mathbf{I}_{QK} + \tilde{\mathbf{A}}\tilde{\mathbf{A}}^T\right) & -\left(\mathbf{I}_{QK} + \tilde{\mathbf{A}}\mathbf{A}^T\right) \\ \mathbf{I}_{QK} + \mathbf{A}\tilde{\mathbf{A}}^T & -\left(\mathbf{I}_{QK} + \mathbf{A}\mathbf{A}^T\right) \end{bmatrix}, \tag{94}$$

(92) can be written as

$$\Phi_D(j\omega) = \frac{1}{|\mathbf{I}_{2QK} - 2j\omega\sigma^2\widehat{\mathbf{G}}|^N} = \prod_{i=1}^{2QK} \frac{1}{|1 - 2j\omega\sigma^2\lambda_i|^N}, \tag{95}$$

where $\lambda_1, \cdots, \lambda_{2QK}$ are the eigenvalues of $\widehat{\mathbf{G}}$. Substituting $z = 2j\omega\sigma^2$, we have

$$\Phi_D(z) = \prod_{i=1}^{2QK} \frac{1}{(1 - z\lambda_i)^M}. \tag{96}$$

From the above characteristic function of D, the PEP in (46) can be obtained as [9],[15]

$$P(\mathbf{b} \rightarrow \tilde{\mathbf{b}}) = -\sum_k \frac{1}{(p_k-1)!}\frac{d^{p_k-1}}{dz^{p_k-1}}\left\{(z-\lambda_k)^{p_k}\frac{\Phi_D(z)}{z}\right\}, \tag{97}$$

where λ_k are the negative eigenvalues of $\tilde{G}$, $Re(\lambda_k) < 0$, and p_k is the multiplicity of λ_k. We obtain (97) in closed-form as follows.

Denote the negative roots as λ_j, with multiplicities $g_j, j = 1, 2, \cdots, L_N$ and the non-negative roots as ρ_i, with multiplicities $r_i, i = 1, 2, \cdots, L_P$, so that $\sum_j g_j + \sum_i r_i = 2QK$. Following similar steps as in [9], we can obtain the closed-form expression for the PEP as

$$P(\mathbf{b} \to \tilde{\mathbf{b}}) = \sum_j \frac{(-\lambda_j)^{N(2K-g_j)}}{\prod_i (\rho_i - \lambda_j)^{Nr_i} \prod_{k \neq j} (\lambda_k - \lambda_j)^{Ng_k}}$$

$$\cdot \sum_{\substack{(l_1,\cdots,l_{Ng_j-1}) \\ 0 \leq l_1,\cdots,l_{Ng_j-1} \leq Ng_j - 1 \\ l_1 + 2l_2 + \cdots + (Ng_j-1)l_{Ng_j-1} = Ng_j - 1}} \prod_{m=1}^{Ng_j-1} \frac{1}{l_m!} \cdot \left[\frac{1}{m} + \frac{N}{m} \left(\sum_i \frac{r_i \rho_i^m}{(\rho_i - \lambda_j)^m} + \sum_{k \neq j} \frac{g_k \lambda_k^m}{(\lambda_k - \lambda_j)^m} \right) \right]^{l_m}. \quad (98)$$

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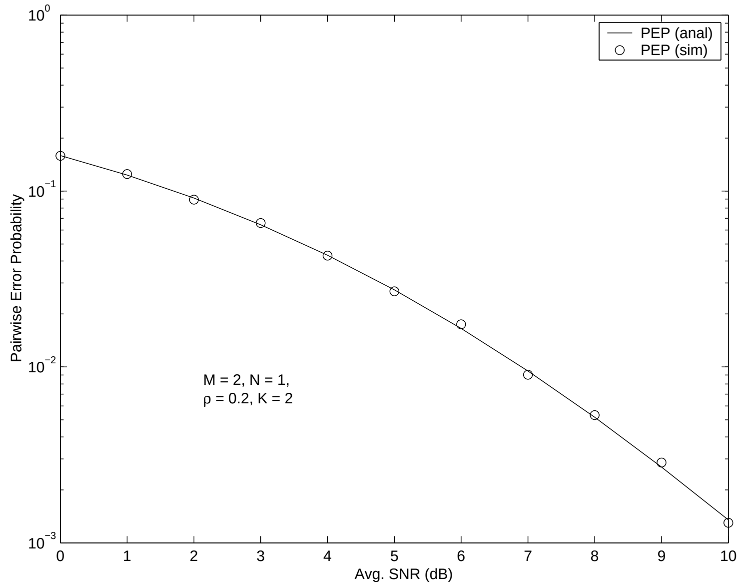

Figure 1: Pairwise error probability as a function of average SNR for $K = 2$, $M = 2$, $N = 1$, $\rho = 0.2$, $\mathbf{b} = (1, 1, 1, -1)^T$, and $\tilde{\mathbf{b}} = (-1, -1, 1, -1)^T$. Case of imperfect channel estimates I. Analysis and simulation.

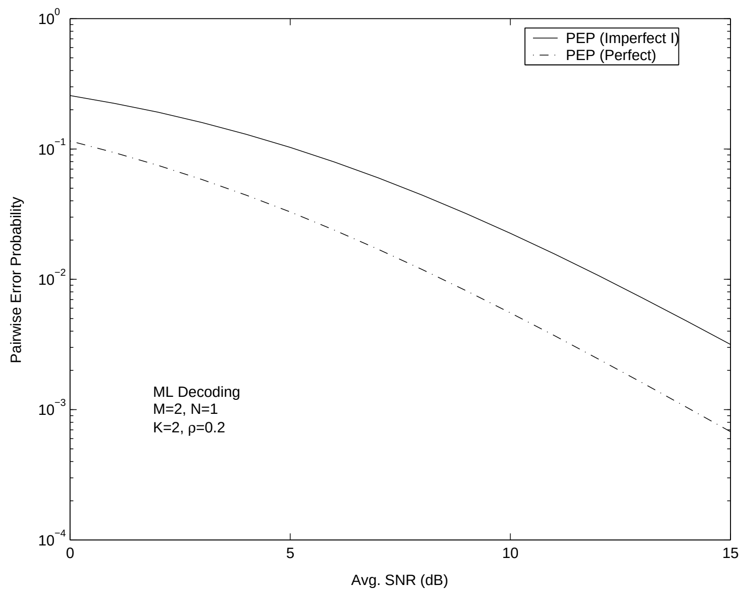

Figure 2: Pairwise error probability as a function of average SNR for $K = 2$, $M = 2$, $N = 1$, $\rho = 0.2$, $\mathbf{b} = (1, 1, 1, 1)^T$, and $\tilde{\mathbf{b}} = (1, 1, -1, -1)^T$. Cases of perfect as well as imperfect channel estimates I. Analysis.

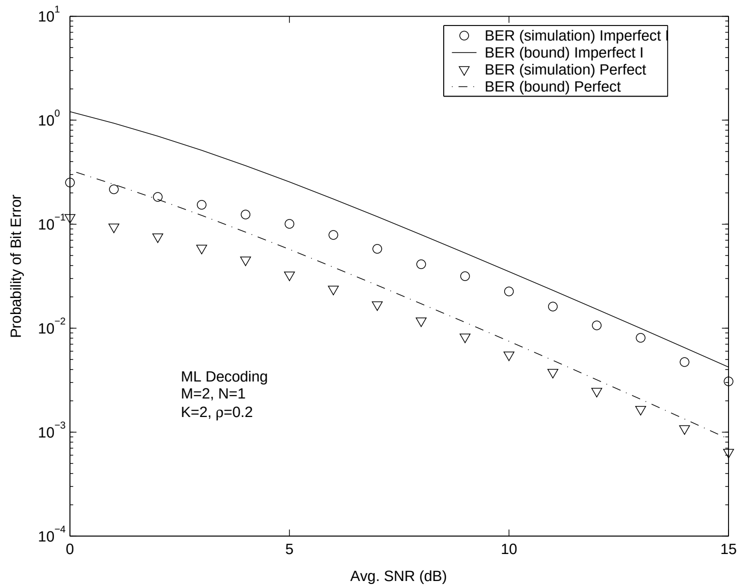

Figure 3: Bit error rate performance as a function of average SNR for $K = 2$, $M = 2$, $N = 1$, and $\rho = 0.2$. Analytical bound as well as simulations. Cases of perfect as well as imperfect channel estimates I.

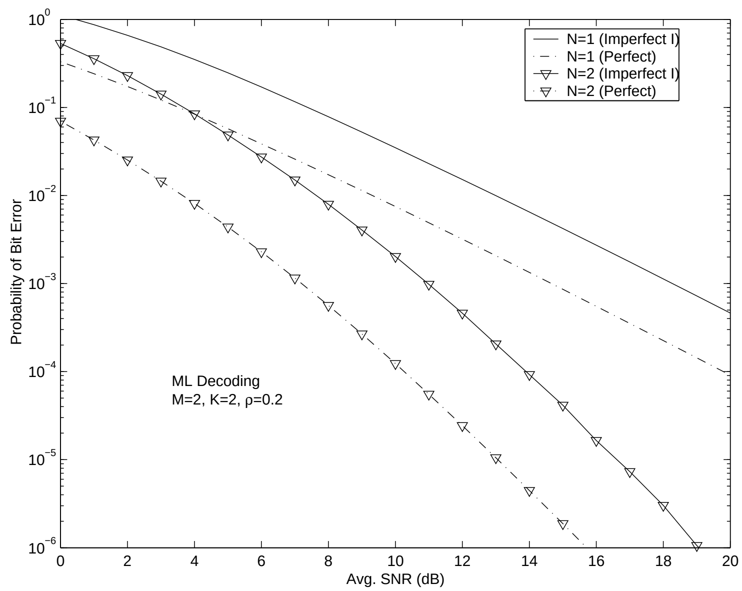

Figure 4: Bit error probability bound as a function of average SNR for different number of Rx antennas, $N = 1, 2$, $K = 2$, $M = 2$, and $\rho = 0.2$. Cases of perfect as well as imperfect channel estimates I. Analysis.

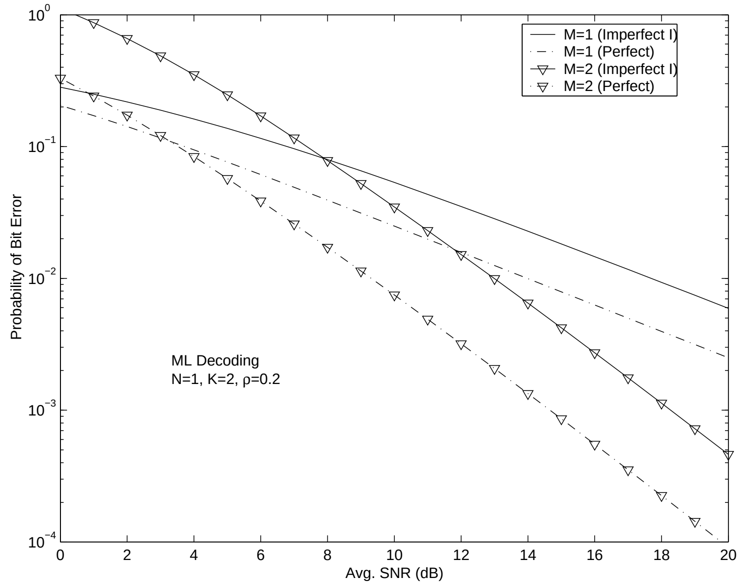

Figure 5: Bit error probability bound as a function of average SNR for different number of Tx antennas, $M = 1, 2$, $K = 2$, $N = 1$, and $\rho = 0.2$. Cases of perfect as well as imperfect channel estimates I. Analysis.

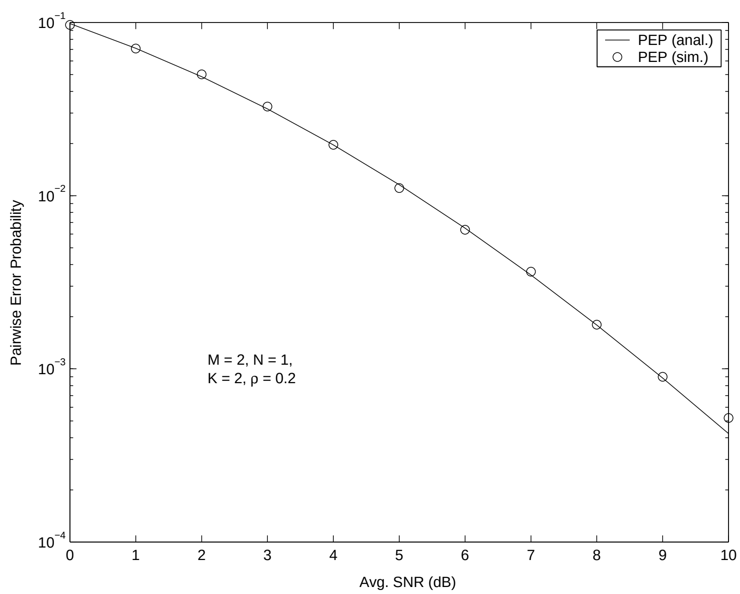

Figure 6: PEP as a function of average SNR for $M = 2$. $K = 2$, $N = 1$, $\rho = 0.2$, $\mathbf{b} = (1, 1, 1, -1)^T$, and $\tilde{\mathbf{b}} = (-1, 1, 1, -1)^T$. Case of imperfect channel estimates II. Analysis and simulations.

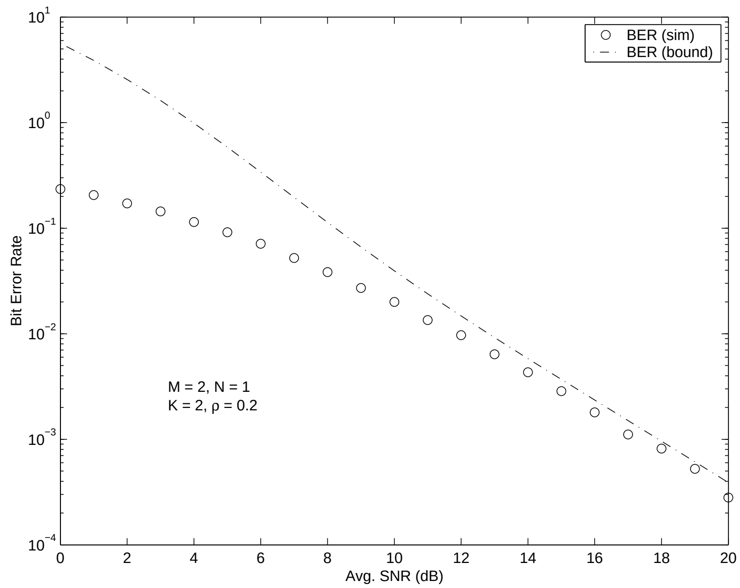

Figure 7: BER as a function of average SNR for $M = 2$. $K = 2$, $N = 1$, and $\rho = 0.2$. Case of imperfect channel estimates II. Analytical BER bound as well as simulations.

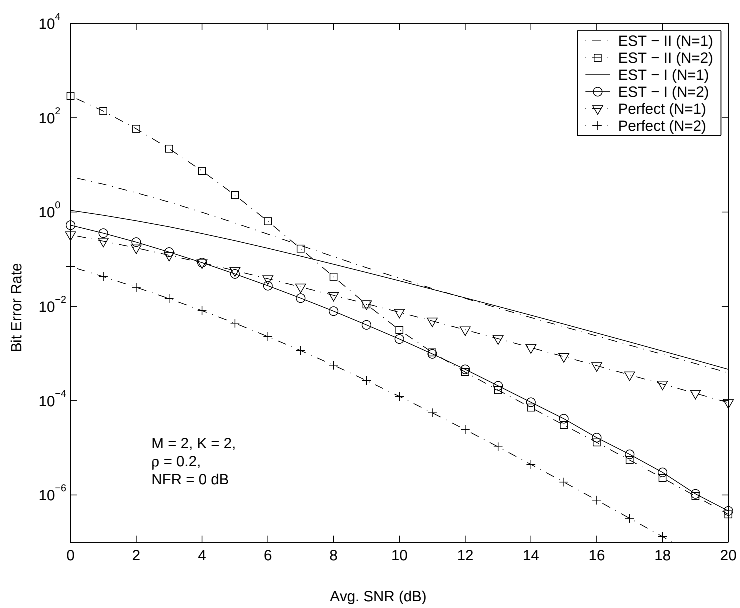

Figure 8: BER bounds as a function of average SNR for $M = 2$. $K = 2$, $N = 1, 2$, and $\rho = 0.2$. Case of perfect as well as imperfect channel estimates I and II. Analysis.