

# Averaging $Q(\|\mathbf{X}\|)$ for a Complex Gaussian Random Vector $\mathbf{X}$ : A Novel Approach

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**Abstract**—In this paper, we compute  $E[Q(\|\mathbf{X}\|)]$ , where  $\mathbf{X}$  is an  $n \times 1$  complex circularly Gaussian vector,  $\|\mathbf{X}\|$  is the  $L^2$  norm of  $\mathbf{X}$  and  $E[\cdot]$  is the expectation operator. This is done by finding the characteristic function of the decision variable and subsequently applying the inversion formula to obtain a one dimensional real integral expression. This integral is then converted to a contour integral which is evaluated using a variant of the Cauchy's integral formula to obtain an expression for  $E[Q(\|\mathbf{X}\|)]$ . We then provide some applications of the above result by obtaining expressions for error probabilities in fading channels.

## I. INTRODUCTION

The average of a Q-function expression is of interest in finding general expressions for the probability of symbol/bit error in slowly fading communication channels, where the argument of the Q-function is a function of a random variable with a well defined probability density function that depends on the kind of fading experienced by the channel [1]. Another application is in finding the bit error probability of the desired user in a fading channel when multiuser detection is being employed at the receiver [2].

The average of a Q-function whose argument is proportional to the square root of a non-central chi-squared random variable with  $2n$  degrees of freedom is obtained by deriving a recursion relation [3]. However, this approach involves integrating the product of the Q-function and the chi-square probability density function. [2]. For the central chi-square distribution, an approach using Craig's formula [4] and the moment generating function (MGF) of the random variable is outlined in [1]. This method is quite complicated but gives the most general closed form expression when the Q-function argument is proportional to the square root of a Nakagami- $m$  [5] random variable.

In this paper, we propose a simple method to average the Q-function whose argument is the  $L^2$  norm of a zero-mean complex circularly Gaussian vector using the characteristic function (CF) of a non-central chi-square distribution [5] and the inversion formula for the cumulative distribution function [6].

## II. PROBLEM STATEMENT

Let  $\mathbf{X}$  be an  $n \times 1$  vector whose entries are complex circularly Gaussian random variables such that

$$\begin{aligned} E[\mathbf{X}] &= \mathbf{m}, \\ E[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^\dagger] &= 2\sigma^2\mathbf{I}_n, \end{aligned} \quad (1)$$

where  $\{\dagger\}$  represents the complex conjugate-transpose operation and  $\mathbf{I}_n$  is the  $n \times n$  identity matrix. Let  $R = \|\mathbf{X}\|^2$ , where  $\|\mathbf{X}\|$  is the  $L^2$  norm of the vector  $\mathbf{X}$ . Then  $R$  is non-central chi-square distributed with  $2n$  degrees of freedom [5] and

$$E[Q(\|\mathbf{X}\|)] = E[Q(\sqrt{R})]. \quad (2)$$

Let  $V$  be a Gaussian random variable with zero mean and unit variance. From [7], we obtain

$$\begin{aligned} E[Q(\sqrt{R})] &= E[P(V > \sqrt{R})] \\ &= \frac{1}{2}P(R - V^2 < 0) \\ &= \frac{1}{2}P(\Delta < 0), \end{aligned} \quad (3)$$

where  $\Delta = R - V^2$ . The characteristic function of  $\Delta$  is given by

$$\begin{aligned} \Phi_\Delta(t) &= E[e^{jt\Delta}] = E[e^{jtR}]E[e^{-jtV^2}] \\ &= \Phi_R(t)\Phi_{V^2}(-t). \end{aligned} \quad (4)$$

Since  $V$  is Gaussian,  $V^2$  chi-square distributed. Hence, we obtain [5]

$$\begin{aligned} \Phi_R(t) &= \frac{e^{\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t}}}{(1-2j\sigma^2 t)^n}, \\ \Phi_{V^2}(-t) &= \frac{1}{(1+2jt)^{\frac{1}{2}}}. \end{aligned} \quad (5)$$

Substituting  $\Phi_R(t)$  and  $\Phi_{V^2}(-t)$  from (5) in (4),

$$\Phi_\Delta(t) = \frac{e^{\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t}}}{(1-2j\sigma^2 t)^n(1+2jt)^{\frac{1}{2}}}. \quad (6)$$

Since

$$\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t} = -\frac{\|\mathbf{m}\|^2}{2\sigma^2} + \frac{\|\mathbf{m}\|^2}{2\sigma^2(1-2j\sigma^2 t)}, \quad (7)$$

(6) can be written as

$$\Phi_\Delta(t) = \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}} e^{\frac{\|\mathbf{m}\|^2}{2\sigma^2(1-2j\sigma^2 t)}}}{(1-2j\sigma^2 t)^n(1+2jt)^{\frac{1}{2}}}. \quad (8)$$

According to the inversion formula of Gil-Pelaez [6], the cumulative distribution function

$$\begin{aligned} F_{\Delta}(x) &= P(\Delta < x) \\ &= \frac{1}{2} + \frac{1}{2\pi j} \int_0^{\infty} \frac{e^{jtx}\Phi_{\Delta}(-t) - e^{-jtx}\Phi_{\Delta}(t)}{t} dt. \end{aligned} \quad (9)$$

From (9), we have

$$\begin{aligned} P(\Delta < 0) &= \frac{1}{2} + \frac{1}{2\pi j} \int_0^{\infty} \frac{\Phi_{\Delta}(-t) - \Phi_{\Delta}(t)}{t} dt \\ &= \frac{1}{2} + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\Phi_{\Delta}(-t)}{t} dt. \end{aligned} \quad (10)$$

From (8) and (10),

$$P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j} \int_{-\infty}^{\infty} \frac{e^{\frac{\|\mathbf{m}\|^2}{2\sigma^2(1+2j\sigma^2t)}}}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2t)^n} dt. \quad (11)$$

Expanding the exponential in the numerator of the integrand in (11) as a power series and interchanging the order of integration and summation, we obtain

$$P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j} \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{\|\mathbf{m}\|^2}{2\sigma^2} \right)^p \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2t)^{n+p}}. \quad (12)$$

It is easy to verify that

$$\frac{1}{t(1+2j\sigma^2t)^n} = \frac{1}{t} - 2j\sigma^2 \sum_{k=1}^n \frac{1}{(1+2j\sigma^2t)^k}. \quad (13)$$

If we let

$$I_n = \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2t)^n}, \quad (14)$$

(12) can be written as

$$P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j} \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{\|\mathbf{m}\|^2}{2\sigma^2} \right)^p I_{n+p} \quad (15)$$

From (13) and (14), we now have

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(1-2jt)^{\frac{1}{2}}} \left[ \frac{1}{t} - 2j\sigma^2 \sum_{k=1}^n \frac{1}{(1+2j\sigma^2t)^k} \right] dt, \quad (16)$$

which, after changing the order of the integral and the summation can be written as

$$I_n = \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}} - 2j\sigma^2 \sum_{k=1}^n \int_{-\infty}^{\infty} \frac{dt}{(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2t)^k}.$$

In the above, letting

$$J = \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}}, \quad (17)$$

$$J_k = 2j\sigma^2 \int_{-\infty}^{\infty} \frac{dt}{(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2t)^k}, \quad (18)$$

we can write (14) as

$$I_n = J - \sum_{k=1}^n J_k. \quad (19)$$

In the next section, we first show that  $J$  and  $J_k$  can be reduced to simple real and contour integrals respectively and then solve them.

### III. SOLVING FOR $I_n$

#### A. The Real Integral

In (17), through a change of variables (from  $t$  to  $-t$ ), we obtain

$$J = - \int_{-\infty}^{\infty} \frac{dt}{t(1+2jt)^{\frac{1}{2}}}. \quad (20)$$

Adding the expressions for  $J$  in (17) and (20),

$$2J = \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}} - \int_{-\infty}^{\infty} \frac{dt}{t(1+2jt)^{\frac{1}{2}}} \quad (21)$$

$$= \int_{-\infty}^{\infty} \frac{1}{t} \left[ \frac{(1+2jt)^{\frac{1}{2}} - (1-2jt)^{\frac{1}{2}}}{(1+4t^2)^{\frac{1}{2}}} \right] dt. \quad (22)$$

Multiplying the numerator and denominator of the integrand in (22) by  $\left[ (1+2jt)^{\frac{1}{2}} - (1-2jt)^{\frac{1}{2}} \right]$ ,

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{t} \left[ \frac{(1+2jt) - (1-2jt)}{(1+4t^2)^{\frac{1}{2}} \left\{ (1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}} \right\}} \right] dt. \quad (23)$$

Cancelling out all common factors,

$$J = 2j \int_{-\infty}^{\infty} \frac{dt}{(1+4t^2)^{\frac{1}{2}} \left\{ (1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}} \right\}}. \quad (24)$$

In the above equation, we note that  $(1-2jt)^{\frac{1}{2}}$  is the complex conjugate of  $(1+2jt)^{\frac{1}{2}}$ . Hence, the integrand in (24) is real as well as an even function of  $t$ . Thus, we get (see Appendix)

$$J = 4j \int_0^{\infty} \frac{dt}{(1+4t^2)^{\frac{1}{2}} \left\{ (1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}} \right\}} \quad (25)$$

$$= j\pi. \quad (26)$$

#### B. The contour integral

The integral in (18) can be solved easily if it can be converted to a contour integral. Towards this end, we state the following Lemma [8].

*Lemma 3.1: Let  $g(x)$  be a function of a real variable  $x$  such that  $|g(x)|$  has a denominator different from zero for all real  $x$  and is of degree in excess of a unit higher than the degree of the numerator. Then*

$$\int_{-\infty}^{\infty} g(x) dx = \int_C g(z) dz, \quad (27)$$

where  $C$  is a semicircle in the complex upper half-plane whose diameter is the real-axis and the integration is in the anti-clockwise sense.

For the integrand in (18),  $k \geq 1$  and the degree of the denominator is greater than that of the numerator by  $k + \frac{1}{2}$ . From Lemma 3.1, we get

$$\begin{aligned} J_k &= 2j\sigma^2 \int_C \frac{dz}{(1-2jz)^{\frac{1}{2}}(1+2j\sigma^2z)^k} \\ &= (2j\sigma^2)^{1-k} \int_C \frac{dz}{(1-2jz)^{\frac{1}{2}}(z - \frac{j}{2\sigma^2})^k}. \end{aligned} \quad (28)$$

We now present a formula for finding the derivatives of an analytic function [8] and subsequently use it to evaluate  $I_k$ .

*Lemma 3.2: If  $g(z)$  is analytic in a domain  $D$ , then it has derivatives of all orders in  $D$  which are then also analytic functions in  $D$ . The value of the  $(k-1)$ th derivative at a point  $z_0$  in  $D$  is given by the formula*

$$g^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi j} \int_L \frac{g(z)}{(z-z_0)^k} \quad (k=1,2,\dots); \quad (29)$$

where  $L$  is any simple closed path in  $D$  which encloses  $z_0$  and whose full interior belongs to  $D$ ; the curve is traversed in the counterclockwise sense and  $g^{(0)}(z_0) = g(z_0)$ , by definition.

The function

$$f(z) = \frac{1}{(1-2jz)^{\frac{1}{2}}} \quad (30)$$

is analytic in the upper half-plane and  $C$  is a closed path in it. Since (28) can be written as

$$J_k = (2j\sigma^2)^{1-k} \int_C \frac{f(z)}{(z - \frac{j}{2\sigma^2})^k} dz, \quad (31)$$

and the point  $\frac{j}{2\sigma^2}$  lies within  $C$ , using Lemma 3.2, we obtain

$$J_k = \frac{2\pi j (2j\sigma^2)^{1-k}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[ \frac{1}{(1-2jz)^{\frac{1}{2}}} \right]_{z=\frac{j}{2\sigma^2}} \quad (32)$$

which, after some simplification, yields

$$\begin{aligned} J_k &= \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}}, \quad k=1, \\ &= \frac{2\pi j\sigma}{(k-1)!} \frac{\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{(2k-3)}{2}}{(1+\sigma^2)^{k-\frac{1}{2}}}, \quad 1 < k \leq n. \end{aligned} \quad (33)$$

Substituting the expressions obtained in (26) and (33) in (19), we get

$$\begin{aligned} I_n &= j\pi - \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}} \left[ 1 + \sum_{k=2}^n \frac{\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{(2k-3)}{2}}{(k-1)!(1+\sigma^2)^{k-1}} \right] \\ &= j\pi - \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}} \left[ 1 + \sum_{k=1}^{n-1} \frac{\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{(2k-1)}{2}}{k!(1+\sigma^2)^k} \right] \\ &= 2\pi j \left[ \frac{1}{2} - \frac{\sigma}{\sqrt{1+\sigma^2}} \sum_{k=0}^{n-1} \binom{2k}{k} \left\{ \frac{1}{4(1+\sigma^2)} \right\}^k \right]. \end{aligned} \quad (34)$$

IV. CLOSED FORM EXPRESSION FOR  $E[Q(\|\mathbf{X}\|)]$   
From (15) and (34),

$$\begin{aligned} P(\Delta < 0) &= 1 - \frac{\sigma e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{\sqrt{1+\sigma^2}} \sum_{p=0}^{\infty} \sum_{k=0}^{n+p-1} \frac{1}{p!} \left( \frac{\|\mathbf{m}\|^2}{2\sigma^2} \right)^p \\ &\quad \times \binom{2k}{k} \left\{ \frac{1}{4(1+\sigma^2)} \right\}^k. \end{aligned} \quad (35)$$

Let  $\alpha = \frac{\|\mathbf{m}\|^2}{2\sigma^2}$  and  $\beta = \frac{1}{1+\sigma^2}$ . Then, changing the indices of summation,

$$P(\Delta < 0) = 1 - e^{-\alpha} \sqrt{1-\beta} (A+B) \quad (36)$$

where<sup>1</sup>

$$A = \sum_{p=0}^{\infty} \sum_{k=0}^p \binom{2k}{k} \left\{ \frac{\beta}{4} \right\}^k \frac{\alpha^p}{p!}, \quad (37)$$

$$B = \sum_{p=0}^{\infty} \sum_{k=p}^{n+p-1} \binom{2k}{k} \left\{ \frac{\beta}{4} \right\}^k \frac{\alpha^p}{p!}. \quad (38)$$

We define the factorial function [9] as

$$(\gamma)_q = \prod_{r=1}^q (\gamma+r-1), \quad (\gamma)_0 = 1, \gamma \neq 0, \quad (39)$$

where  $q$  is a positive integer.

A. The  $B$  series

Since

$$\binom{2k}{k} = \frac{4^k \left(\frac{1}{2}\right)_k}{k!}, \quad (40)$$

we obtain

$$B = \sum_{p=0}^{\infty} \sum_{k=p}^{n+p-1} \frac{\left(\frac{1}{2}\right)_k \beta^k \alpha^p}{k! p!}. \quad (41)$$

Changing the limits of summation in (41),

$$\begin{aligned} B &= \sum_{p=0}^{\infty} \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_{k+p} \beta^{k+p} \alpha^p}{(k+p)! p!} \\ &= \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \sum_{p=0}^{\infty} \frac{\left(\frac{1}{2}+k\right)_p (\alpha\beta)^p}{(k+1)_p p!} \\ &= \sum_{k=0}^{n-1} \binom{2k}{k} \left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2}+k; k+1; \alpha\beta\right), \end{aligned} \quad (42)$$

where  ${}_1F_1(a; b; x)$  is the confluent hypergeometric function [9]. According to Kummer's formula for the confluent hypergeometric function,

$${}_1F_1(a; b; x) = e^x {}_1F_1(b-a; b; -x). \quad (43)$$

Using this result in (42), we obtain

$$B = \exp(\alpha\beta) \sum_{k=0}^{n-1} \binom{2k}{k} \left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2}; k+1; -\alpha\beta\right). \quad (44)$$

<sup>1</sup>We assume that all the infinite series considered henceforth converge.

### B. The A series

We rewrite (37) as

$$A = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k \alpha^p}{k! p!} - \sum_{p=0}^{\infty} \sum_{k=p+1}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k \alpha^p}{k! p!} \quad (45)$$

In the above,

$$\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k \alpha^p}{k! p!} = \left( \sum_{p=0}^{\infty} \frac{\alpha^p}{p!} \right) \left( \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \right). \quad (46)$$

Since  $|\beta| < 1$ , the second sum on the right hand side of (46) is the binomial series, i.e.,

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} = (1 - \beta)^{-\frac{1}{2}}. \quad (47)$$

Thus,

$$A = \frac{e^{\alpha}}{\sqrt{1 - \beta}} - S, \quad (48)$$

where

$$\begin{aligned} S &= \sum_{p=0}^{\infty} \sum_{k=p+1}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k \alpha^p}{k! p!} \\ &= \sum_{p=0}^{\infty} \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+p} \beta^{k+p} \alpha^p}{(k+p)! p!}. \end{aligned} \quad (49)$$

Following the steps in (42), (49) can be written as

$$S = \sum_{k=1}^{\infty} \binom{2k}{k} \left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2} + k; k + 1; \alpha\beta\right). \quad (50)$$

The above infinite series has a closed form expression [3]

$$S = \frac{2 \exp\left(\frac{\alpha\beta}{2}\right)}{\sqrt{1 - \beta}} \quad (51)$$

$$\times \left[ \exp\left\{\frac{\alpha}{2}(1 + \beta)\right\} Q_1(u, w) - \frac{1}{2}(1 + \sqrt{1 - \beta}) I_0\left(\frac{\alpha\beta}{2}\right) \right], \quad (52)$$

where

$$\begin{aligned} u &= \sqrt{\frac{\alpha}{2}(2 - \beta) - \frac{2}{\beta}\sqrt{1 - \beta}} \\ w &= \sqrt{\frac{\alpha}{2}(2 - \beta) + \frac{2}{\beta}\sqrt{1 - \beta}}, \end{aligned} \quad (53)$$

and  $Q_1(u, w)$  is the Marcum Q-function [5]. Substituting (51) in (48) gives us a closed form expression for  $A$ . Since we already have a compact expression for  $B$  in (44), replacing the infinite series for  $A$  and  $B$  in (35) by their respective closed form expressions, and noting from (2) and (3) that

$$E[Q(\|\mathbf{X}\|)] = \frac{1}{2} P(\Delta < 0), \quad (54)$$

we obtain an exact expression for  $E[Q(\|\mathbf{X}\|)]^2$ .

*Corollary:* When  $\mathbf{X}$  is zero mean, from (34) and (35), we obtain

<sup>2</sup>Lindsey, in [3], using a fundamentally different approach, has obtained a closed form expression in a completely different form.

$$\begin{aligned} P(\Delta < 0) &= \frac{1}{2} + \frac{I_n}{2\pi j} \\ &= 1 - \frac{\sigma}{\sqrt{1 + \sigma^2}} \sum_{k=0}^{n-1} \binom{2k}{k} \left\{ \frac{1}{4(1 + \sigma^2)} \right\}^k. \end{aligned} \quad (55)$$

Substituting the above in (54) leads to the well known result [2]

$$E[Q(\|\mathbf{X}\|)] = \frac{1}{2} \left[ 1 - \frac{\sigma}{\sqrt{1 + \sigma^2}} \sum_{k=0}^{n-1} \binom{2k}{k} \left\{ \frac{1}{4(1 + \sigma^2)} \right\}^k \right] \quad (56)$$

### V. EXAMPLE: AVERAGE PROBABILITY OF ERROR FOR NAKAGAMI- $m$ FADING CHANNELS

If a random variable  $\alpha$  is Nakagami- $m$  distributed, the random variable  $\gamma = \frac{\alpha^2 \varepsilon_b}{N_0}$  has the probability density function [5]

$$p(\gamma) = \frac{m^m}{\Gamma(m)\bar{\gamma}} \gamma^{m-1} e^{-m\gamma/\bar{\gamma}}, \quad (57)$$

where  $\bar{\gamma} = \frac{E(\alpha^2)\varepsilon_b}{N_0}$ . For fading channels, the average probability of error is given by

$$P_e = \int_0^{\infty} Q(a\sqrt{\gamma}) p_{\gamma}(\gamma) d\gamma, \quad (58)$$

where  $a$  is a constant that depends on the specific modulation/detection combination [1]. We note that  $\gamma$  has the same distribution as  $R$  (when  $\mathbf{X}$  has zero mean) with  $\sigma^2 = \frac{\bar{\gamma}}{2m}$  when  $m$  is an integer [5]. Thus, after accounting for the constant  $a$  in (58), the average probability of error for a Nakagami- $m$  fading channel is obtained as

$$P_e = \frac{1}{2} \left[ 1 - \sqrt{\frac{a^2 \bar{\gamma}}{2m + a^2 \bar{\gamma}}} \sum_{k=0}^{m-1} \binom{2k}{k} \left\{ \frac{2m}{4(2m + a^2 \bar{\gamma})} \right\}^k \right] \quad (59)$$

by substituting  $\sigma^2 = \frac{a^2 \bar{\gamma}}{2m}$  and replacing  $n$  by  $m$  in (56). We note that exactly the same result has been arrived at using a different approach in [1], equation (5.18).

### APPENDIX

Let  $1 + 2jt = re^{j\theta}$ , where  $r = (1 + 4t^2)^{\frac{1}{2}}$  and  $\cos \theta = \frac{1}{r}$ . Since

$$(1 + 2jt)^{\frac{1}{2}} + (1 - 2jt)^{\frac{1}{2}} = 2r^{\frac{1}{2}} \cos \frac{\theta}{2}, \quad (60)$$

and

$$\begin{aligned} \cos \frac{\theta}{2} &= \sqrt{\frac{1}{2}(1 + \cos \theta)} \\ &= \sqrt{\frac{1}{2}\left(1 + \frac{1}{r}\right)}, \end{aligned} \quad (61)$$

the integrand in (25)

$$\frac{1}{(1 + 4t^2)^{\frac{1}{2}} \left\{ (1 + 2jt)^{\frac{1}{2}} + (1 - 2jt)^{\frac{1}{2}} \right\}} = \frac{1}{2r\sqrt{\frac{1+r}{2}}}. \quad (62)$$

Hence,

$$\begin{aligned} J &= 4j \int_0^\infty \frac{dt}{(1+4t^2)^{\frac{1}{2}} \left\{ (1+2jt)^{\frac{1}{2}} + (1-2jt)^{\frac{1}{2}} \right\}} \quad (63) \\ &= 2\sqrt{2}j \int_0^\infty \frac{dt}{(1+4t^2)^{\frac{1}{2}} \left\{ 1 + (1+4t^2)^{\frac{1}{2}} \right\}^{\frac{1}{2}}}. \end{aligned}$$

Substituting  $t = \frac{\tan \phi}{2}$  in the above,

$$J = \sqrt{2}j \int_0^{\frac{\pi}{2}} \frac{\sec \phi}{\sqrt{1 + \sec \phi}} d\phi. \quad (64)$$

Let  $\sqrt{1 + \sec \phi} = \sqrt{2} \sec \psi$ . Then

$$\frac{\sec \phi}{\sqrt{1 + \sec \phi}} d\phi = \sqrt{2} d\psi. \quad (65)$$

The integral in (64) now becomes

$$\begin{aligned} J &= 2j \int_0^{\frac{\pi}{2}} d\psi \quad (66) \\ &= j\pi. \end{aligned}$$

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