Outage Probability of Cellular Mobile Radio in the Presence of Multiple Nakagami Interferers with Similar Fading Parameters : A Correction

G. V. V. Sharma
SPANN Lab, EE Department,
IIT Bombay, INDIA 400076
Email: gadepall@gmail.com

***Abstract*— Previous available results for the probability of outage with multiple Nakagami interferers with non-integer fading parameters have been derived using the characteristic function approach. This was done by converting an improper real integral, whose integrand has a singularity at the origin, to a contour integral, which was then evaluated using the method of residues. However, the method is mathematically valid only when the real integral exists. In this paper, we show that the existence of the real integral has not been established and the earlier approach is therefore incorrect. For the special case of multiple Nakagami interferers with similar non-integer fading parameters, using a slightly different but rigorous approach, we find an exact expression for the probability of outage.**

***Index Terms*— Nakagami fading, inversion formula, Cauchy principal value**

I. Introduction

A characteristic function approach has been suggested by Zhang in [1] to compute the outage probability in a cellular network with multiple Nakagami interferers having arbitrary fading parameters. This method involves the transformation of a real improper integral to a complex integral. The complex integral is then evaluated using the residue theorem, which gives the Cauchy principal value (c.p.v) of the real integral [2]. When the integral exists, its value is equal to the c.p.v [3]. However, the integrand in [1] has a singularity at the origin, which means that the existence of the integral has to be proved before the contour integral approach can be used.

In this paper, we show how the c.p.v of the integral has been used to compute the probability of outage in [1], without proving the existence of the integral. Then, for the case of multiple Nakagami interferers with similar non-integer fading parameters, we propose a slightly different approach, where we express the integral as the sum of two complex integrals choosing different contours for integrating the two, establishing the convergence of the integrals wherever necessary. Cauchy's integral formula for analytic functions [2] is then used to evaluate these integrals to obtain an expression for the outage probability.

The rest of the paper is organized as follows. In Section II, we introduce the system model and present the formula for calculating the outage probability in terms of a real integral involving the characteristic function. Then the flaw in the process of transforming the real integral to the corresponding contour integral in [1] is described in Section III. In Section IV, the modified approach to evaluate the outage probability for the special case of Nakagami interferers with similar fading parameters is presented. Conclusions are available in Section V.

II. System Model

We follow the notation in [1], with $r_0(t)$ being the amplitude of the desired signal received at the mobile unit and $r_k(t), \quad k = 1, \ldots, L$ are the amplitudes of the L co-channel Nakagami interferers. The probability density function (PDF) of $\xi_k = r_k^2(t)$ follows the Gamma distribution and is given by

$$f_{\xi_k}(y) = (\lambda_k)^{m_k} \frac{y^{m_k-1}}{\Gamma(m_k)} \exp(-\lambda_k y), \quad y \geq 0, \quad k = 0, \cdots, L \tag{1}$$

where m_k is the fading parameter with values ranging from $[0.5, \infty)$ and m_0 is a positive integer. Also,

$$\lambda_k = \frac{m_k}{\Omega_k} \tag{2}$$

where $\Omega_k > 0$ is the average power of $r_k(t)$. The mean and variance of ξ_k are

$$E[\xi_k] = \Omega_k,$$
$$Var[\xi_k] = \frac{\Omega_k^2}{m_k}. \tag{3}$$

In the above scenario, outage occurs in the event of $q\sum_{k=1}^{L} \xi_k > \xi_0$, where q is the prescribed power protection ratio. We define the random variable

$$\gamma = q \sum_{k=1}^{L} \xi_k - \xi_0. \tag{4}$$

Hence, the probability of outage $P_{out} = P(\gamma > 0)$. The characteristic function of the random variable γ is given by [1]

$$\phi_\gamma(t) = \frac{1}{\prod_{k=1}^{L}(1 - \frac{jqt}{\lambda_k})^{m_k}(1 + \frac{jt}{\lambda_0})^{m_0}}. \tag{5}$$

Then, using the Gil-Pelaez theorem [4],

$$P_{out} = \frac{1}{2} + \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t} dt. \tag{6}$$

We define

$$I_\gamma = \int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t} dt. \tag{7}$$

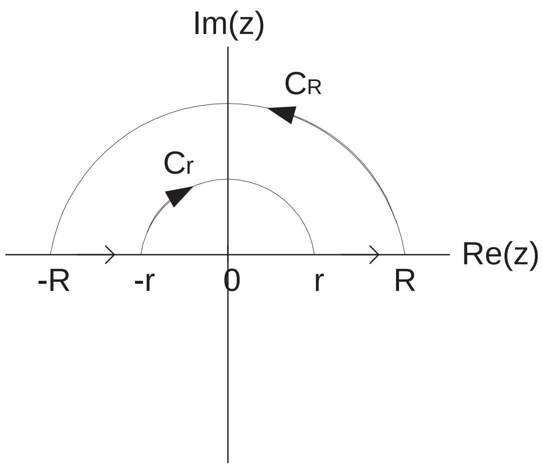

Fig. 1. The contour C.

III. The contour integral

Following the method outlined in [1], we consider the closed path C in the complex plane in Fig. 1. Let C_R and C_r be two semi-circular paths with radii R and r respectively. The contour integral

$$\int_C \frac{\phi_\gamma(z)}{z}dz = \int_{C_R} \frac{\phi_\gamma(z)}{z}dz + \int_{-R}^{-r} \frac{\phi_\gamma(t)}{t}dt + \int_{C_r} \frac{\phi_\gamma(z)}{z}dz + \int_{r}^{R} \frac{\phi_\gamma(t)}{t}dt. \quad (8)$$

Since the denominator of $\frac{\phi_\gamma(z)}{z}$ is at least two units greater than the numerator, we obtain [3]

$$\lim_{R\to\infty} \int_{C_R} \frac{\phi_\gamma(z)}{z}dz = 0. \quad (9)$$

Also, it has been shown in [1] that

$$\lim_{r\to 0} \int_{C_r} \frac{\phi_\gamma(z)}{z}dz = -j\pi. \quad (10)$$

From (8), (9) and (10), we get

$$\lim_{\substack{r\to 0\\ R\to\infty}} \int_{-R}^{-r} \frac{\phi_\gamma(t)}{t}dt + \int_{r}^{R} \frac{\phi_\gamma(t)}{t}dt = j\pi + \int_C \frac{\phi_\gamma(z)}{z}. \quad (11)$$

Since m_0 is an integer, $\phi_\gamma(z)$ has multiple poles of order m_0 at $z = j\lambda_0$, which lies in the upper half plane inside the contour C. Hence, using the residue theorem [2],

$$\int_C \frac{\phi_\gamma(z)}{z} = \frac{2\pi j}{(m_0-1)!}\frac{d^{m_0-1}}{dz^{m_0-1}}\left[\frac{\phi_\gamma(z)}{z}\right]_{z=j\lambda_0}. \quad (12)$$

Definition: The Cauchy principal value (c.p.v) of a definite integral

$$\int_A^B f(t)dt \quad (13)$$

whose integrand becomes infinite at a point a in the interval of integration, i.e.

$$\lim_{t\to a} |f(t)| = \infty \quad (14)$$

is defined as [2]

$$c.p.v \int_A^B f(t)dt = \lim_{\epsilon\to 0}\left[\int_A^{a-\epsilon} f(t)dt + \int_{a-\epsilon}^B f(t)dt\right]. \quad (15)$$

The integral itself is defined as

$$\int_A^B f(t)dt = \lim_{\epsilon\to 0}\int_A^{a-\epsilon} f(t)dt + \lim_{\eta\to 0}\int_{a-\eta}^B f(t)dt \quad (16)$$

where both ϵ and η approach zero through positive values. It may so happen that neither of the two limits in (16) exist, i.e. that the integral itself has no meaning but the c.p.v defined by (15) exists. This leads to the following lemma [3].

Lemma 3.1: When a definite integral has a singularity in the interval of integration and the c.p.v exists, then, if the integral exists, the c.p.v is equal to the value of the integral.

Using the definition of the c.p.v in (15), from (11), we obtain

$$\begin{aligned} c.p.v \int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t}dt &= \lim_{\substack{r\to 0\\ R\to\infty}} \int_{-R}^{-r} \frac{\phi_\gamma(t)}{t}dt + \int_r^R \frac{\phi_\gamma(t)}{t}dt \\ &= j\pi + \int_C \frac{\phi_\gamma(z)}{z}. \quad (17) \end{aligned}$$

Here, we note that (17) has been used to evaluate I_γ in [1], but the existence of the integral itself has not been established. A special case when the integral actually does not converge is given in the following example.

Example 1: Let $L = 1, m_0 = 1, m_1 = 1$ and $\frac{\lambda_1}{\lambda_0} = q$. This choice of q may not be practical, but since we are challenging the mathematical validity of the approach in [1], we are justified in using it. Then

$$\phi_\gamma(t) = \frac{1}{1+\frac{t^2}{\lambda_0^2}}. \quad (18)$$

Hence,

$$\begin{aligned} I_\gamma &= \int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t}dt \quad (19) \\ &= \int_{-\infty}^{\infty} \frac{dt}{t\left(1+\frac{t^2}{\lambda_0^2}\right)} \\ &= \int_{-\infty}^{\infty} \frac{dt}{t\,(1+t^2)} \quad (20) \end{aligned}$$

after appropriate substitutions. Since

$$\frac{1}{t\,(1+t^2)} = \frac{1}{t} - \frac{t}{1+t^2}, \quad (21)$$

we obtain

$$I_\gamma = \int_{-\infty}^{\infty} \frac{dt}{t} - \int_{-\infty}^{\infty} \frac{t}{1+t^2}dt. \quad (22)$$

From elementary calculus, it is obvious that neither of the integrals on the right hand side of (22) converge, so I_γ does not converge in this case. However, using (15), it is easy to see that

$$c.p.v \quad I_\gamma = 0, \quad (23)$$

which can also be verified using (17).

Thus, the actual relation between P_{out} and $\phi_\gamma(t)$ in [1] is

$$\begin{aligned} P_{out} &= \frac{1}{2} + \frac{1}{2\pi j} \times c.p.v \quad I_\gamma \\ &= \frac{1}{2} + \frac{1}{2\pi j} \times c.p.v \int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t}dt, \quad (24) \end{aligned}$$

which is different from (6) when I_γ does not exist, as in Example 1.

IV. Interferers with similar fading parameters

For the special case when $m_k = m$ and $\lambda_k = \lambda$, from (5), we obtain

$$\phi_\gamma(t) = \frac{1}{(1-\frac{jqt}{\lambda})^{mL}(1+\frac{jt}{\lambda_0})^{m_0}}. \quad (25)$$

With an appropriate substitution of variables,

$$\begin{aligned} I_\gamma &= \int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t} dt \\ &= \int_{-\infty}^{\infty} \frac{dt}{t(1-\frac{jq\lambda_0 t}{\lambda})^{mL}(1+jt)^{m_0}}. \end{aligned} \quad (26)$$

Since

$$\frac{1}{t(1+jt)^{m_0}} = \frac{dt}{t} - j\sum_{n=1}^{m_0} \frac{1}{(1+jt)^n}, \quad (27)$$

substituting $\sigma = \frac{q\lambda_0}{\lambda}$, (26) becomes

$$I_\gamma = \int_{-\infty}^{\infty} \frac{dt}{t(1-j\sigma t)^{mL}} - j\sum_{n=1}^{m_0} \int_{-\infty}^{\infty} \frac{dt}{(1-j\sigma t)^{mL}(1+jt)^n}, \quad (28)$$

Now, let

$$J = \int_{-\infty}^{\infty} \frac{dt}{t(1-j\sigma t)^{mL}} \quad (29)$$

and

$$J_n = j\int_{-\infty}^{\infty} \frac{dt}{(1-j\sigma t)^{mL}(1+jt)^n}. \quad (30)$$

A. The integral J_n

The integrand in (30) has multiple poles at j in the upper half plane and no singularities on the real line. Further, the degree of the denominator is more than a unit greater than that of the numerator. Thus, it can be converted to a contour integral [3]

$$J_n = j\int_S \frac{dz}{(1-j\sigma z)^{mL}(1+jz)^n}, \quad (31)$$

where S is a semi-circle of infinite radius. Applying the residue theorem, we obtain

$$J_n = \frac{2\pi j}{(n-1)! j^{n-1}} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{1}{(1-j\sigma z)^{mL}}\right]_{z=j}, n = 1, \ldots, m_0 \quad (32)$$

The above expression admits a closed form

$$J_n = 2\pi j \frac{\sigma^{n-1}(mL)_{n-1}}{(n-1)!(1+\sigma)^{mL+n-1}}, \quad (33)$$

where the *factorial function* $(\gamma)_q$ is defined as

$$(\gamma)_q = \prod_{r=1}^{q} (\gamma + r - 1), (\gamma)_0 = 1, \gamma \neq 0, \quad (34)$$

q being a positive integer.

B. The integral J

The integrand in J has a singularity at the origin. Hence we choose the contour C in Fig. 1, and since the degree of the denominator of the integrand in J is more than a unit greater than that of the numerator, we obtain from (17)

$$c.p.v \quad J \quad = \quad j\pi + \int_C \frac{dz}{z(1-j\sigma z)^{mL}}. \quad (35)$$

Since the integrand in J is analytic in the region enclosed by the contour C, we have, from Cauchy's integral formula [2]

$$\int_C \frac{dz}{z(1-j\sigma z)^{mL}} = 0. \quad (36)$$

Thus,

$$c.p.v \quad J \quad = \quad j\pi. \quad (37)$$

By a change of variables (from t to $-t$) in (29), we obtain

$$J = -\int_{-\infty}^{\infty} \frac{dt}{t(1+j\sigma t)^{mL}}. \quad (38)$$

Adding (29) and (38)

$$2J \quad = \quad \int_{-\infty}^{\infty} \frac{dt}{t}\left[\frac{1}{(1-j\sigma t)^{mL}} - \frac{1}{(1+j\sigma t)^{mL}}\right]. \quad (39)$$

From (39), we have

$$2J \quad = \quad \int_{-\infty}^{\infty} \frac{dt}{t}\left[\frac{(1+j\sigma t)^{mL} - (1-j\sigma t)^{mL}}{(1+\sigma^2 t^2)^{mL}}\right]. \quad (40)$$

In the above, substituting $\tan\theta = \sigma t$,

$$2J = 2j\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2\theta}{\tan\theta} \frac{\sin(mL\theta)}{\sec^{mL}\theta} d\theta, \quad (41)$$

which gives us

$$J = j\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(mL\theta)}{\sin\theta} \cos^{mL-1}\theta \quad d\theta. \quad (42)$$

The singularity at $t = 0$ in (29), which is reflected at $\theta = 0$ in (42), vanishes due to the expression $\frac{\sin(mL\theta)}{\sin\theta}$ in the integrand. Thus, the integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin(mL\theta)}{\sin\theta} \cos^{mL-1}\theta \quad d\theta \quad (43)$$

has an integrand that is continuous in the interval of integration and is non-singular at the origin. This is sufficient to conclude that the above integral exists [3]. Hence, the integral J exists. We note that this happens because the interferers have similar fading parameters. From Lemma 3.1 and (37),

$$J = j\pi. \quad (44)$$

From (6),(28), (33) and (44), after substituting for σ, we obtain the outage probability for multiple Nakagami interferers with similar non-integer fading parameters

$$P_{out} = 1 - \sum_{n=0}^{m_0-1} \frac{(q\lambda_0)^n (mL)_n}{n!(\lambda + q\lambda_0)^{mL+n}}. \quad (45)$$

V. CONCLUSIONS

The real improper integral obtained using [4] to evaluate the outage probability in [1] was assumed to converge in general. We have shown that this is not true by providing a counter example, thus concluding that the approach in [1] is not mathematically rigorous. We then provided an alternative method for evaluating the outage probability for the special case of Nakagami interferers having similar non-integer fading characteristics.

REFERENCES

[1] Q. T. Zhang, "Outage probability of cellular mobile radio in the presence of multiple Nakagami interferers with arbitrary fading parameters," *IEEE Trans. Veh. Tech.*, vol. 44, no.3, August 1995.

[2] E. Kreyszig, *Advanced engineering mathematics*, 5th ed. : John Wiley and Sons, 1983.

[3] J. E. Marsden, *Basic complex analysis*, W. H. Freeman and Company, 1973.

[4] J. Gil-Pelaez, "Note on the inversion theorem," *Biometrika*, vol. 38, 1951, pp. 481-482.