Averaging $Q(\|\mathbf{X}\|)$ for a Complex Circularly Gaussian Random Vector $\mathbf{X}$: A Novel Approach

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***Abstract*— In this paper, we average $Q(\|\mathbf{X}\|)$, where $\|\mathbf{X}\|$ is the Euclidean norm of an $n \times 1$ complex circularly Gaussian vector $\mathbf{X}$. This is done by finding the characteristic function of the decision variable and subsequently applying the Gil-Pelaez theorem to obtain a one dimensional real integral. The integral is then converted to a contour integral which is evaluated using a variant of the Cauchy's integral formula to obtain an expression for $E[Q(\|\mathbf{X}\|)]$, where $E[\ ]$ is the expectation operator.**

I. Introduction

The average of a Q-function expression is of interest in finding general expressions for the probability of symbol error in slowly fading communication channels, where the argument of the Q-function is a function of a random variable with a well defined probability density function that depends on the kind of fading experienced by the channel [1].

In [2], the evaluation of the performance of self-adaptive systems over a Rayleigh faded multipath channel involves averaging a Q-function expression whose argument is the square root of the sum of the squares of a set of independent and identically distributed (i.i.d) Rayleigh distributed random variables. The process is repeated for Rician random variables in the classic work by Lindsey [3] to evaluate the error probability of multichannel reception of binary signals in Rician fading.

For averaging the Q-function involving Rayleigh random variables, alternative techniques such as the Craig's formula based approach for evaluating the probability of error for Nakagami-m fading channels [1] are available. For the case of Rician random variables, Lindsey's approach is the only one that we could find in the available literature.

In this correspondence, we suggest a characteristic function based approach to find a closed form expression for the average of a Q-function whose argument is the norm of a complex circularly Gaussian random vector. This is done by using the method outlined in [8] and the inversion formula for the cumulative distribution function [7] to obtain a real integral involving the characteristic function of the decision variable. This integral is then converted to a contour integral, which is then evaluated using the Cauchy's integral formula [9] for multiple poles. When the mean of the Gaussian vector is non-zero, we obtain an infinite series which can be expressed in closed form using special functions.

Since the square root of the sum of the squares of Rayleigh and Rician distributed random variables with similar variances is the norm of a complex circularly Gaussian random vector with zero and non-zero mean respectively, our results can be used to find alternative solutions to the problems considered in [2] and [3]. Another application is in evaluating the performance of decorrelating multiuser detectors in fading channels [4].

Lindsey [3] uses the direct probability density function based approach to first develop a recursive equation for the probability of symbol error. Then, the initial condition is derived. Both steps involve manipulating complicated integrals with special functions, like the higher order Bessel functions, as integrands. A thorough knowledge of special functions is required to understand this approach.

The novelty of our work lies in the characteristic function based approach, where we convert the real integral to a contour integral, which is easily evaluated using the method of residues [9]. As a result, instead of evaluating complicated integrals involving Bessel functions as in [3], the problem reduces to finding closed form expressions for certain infinite series. As will be obvious later, this is relatively easy and makes our approach much simpler compared to [3].

II. Problem Statement

Let $\mathbf{X}$ be an $n \times 1$ complex circularly Gaussian vector defined by

$$\begin{aligned} E[\mathbf{X}] &= \mathbf{m}, \\ E[(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^T] &= \mathbf{0}, \\ E[(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^\dagger] &= 2\sigma^2 \mathbf{I}_n, \end{aligned} \tag{1}$$

where $\{^T\}$ represents the transpose operation, $\{^\dagger\}$ represents the complex conjugate-transpose and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Let $R = \|\mathbf{X}\|^2$, where $\|\mathbf{X}\| = \sqrt{\mathbf{X}^\dagger \mathbf{X}}$ is the Euclidean norm of the vector $\mathbf{X}$. Then R is non-central chi-square distributed with $2n$ degrees of freedom [6] and

$$E[Q(\|\mathbf{X}\|] = E[Q(\sqrt{R})]. \tag{2}$$

Let V be a Gaussian random variable with zero mean and unit variance. The $Q()$ function in (2) is then defined as $P(V > x), x > 0$, i.e.,

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{\frac{-y^2}{2}} dy, \quad x > 0 \tag{3}$$

From [8], we obtain

$$\begin{aligned} E[Q(\sqrt{R})] &= E[P(V > \sqrt{R})] \\ &= \frac{1}{2}P(R - V^2 < 0) \\ &= \frac{1}{2}P(\Delta < 0), \end{aligned} \tag{4}$$

where $\Delta = R - V^2$. Since R and V are independent, the characteristic function of Δ is given by

$$\begin{aligned} \Phi_\Delta(t) &= E[e^{jt\Delta}] = E[e^{jtR}]E[e^{-jtV^2}] \\ &= \Phi_R(t)\Phi_{V^2}(-t). \end{aligned} \tag{5}$$

Since V is Gaussian, V^2 is chi-square distributed. Hence, we obtain [6]

$$\begin{aligned} \Phi_R(t) &= \frac{e^{\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t}}}{(1-2j\sigma^2 t)^n}, \\ \Phi_{V^2}(-t) &= \frac{1}{(1+2jt)^{\frac{1}{2}}}. \end{aligned} \tag{6}$$

Substituting $\Phi_R(t)$ and $\Phi_{V^2}(-t)$ from (6) in (5),

$$\Phi_\Delta(t) = \frac{e^{\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t}}}{(1-2j\sigma^2 t)^n(1+2jt)^{\frac{1}{2}}}. \tag{7}$$

Since

$$\frac{j\|\mathbf{m}\|^2 t}{1-2j\sigma^2 t} = -\frac{\|\mathbf{m}\|^2}{2\sigma^2} + \frac{\|\mathbf{m}\|^2}{2\sigma^2(1-2j\sigma^2 t)}, \tag{8}$$

(7) can be written as

$$\Phi_\Delta(t) = \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}} e^{\frac{\|\mathbf{m}\|^2}{2\sigma^2(1-2j\sigma^2 t)}}}{(1-2j\sigma^2 t)^n(1+2jt)^{\frac{1}{2}}}. \tag{9}$$

According to the general form[1] of the inversion formula of Gil-Pelaez [7], the cumulative distribution function

$$\begin{aligned} F_\Delta(x) &= P(\Delta < x) \\ &= \frac{1}{2} + \frac{1}{2\pi j} \times \lim_{\substack{\epsilon\to 0\\ \lambda\to\infty}} \int_\epsilon^\lambda \frac{e^{jtx}\Phi_\Delta(-t) - e^{-jtx}\Phi_\Delta(t)}{t} dt. \end{aligned} \tag{10}$$

From (10), we have

$$\begin{aligned} P(\Delta < 0) &= \frac{1}{2} + \frac{1}{2\pi j} \times \lim_{\substack{\epsilon\to 0\\ \lambda\to\infty}} \int_\epsilon^\lambda \frac{\Phi_\Delta(-t) - \Phi_\Delta(t)}{t} dt \\ &= \frac{1}{2} + \frac{1}{2\pi j} \times c.p.v \int_{-\infty}^{\infty} \frac{\Phi_\Delta(-t)}{t} dt. \end{aligned} \tag{11}$$

where the Cauchy principal value (c.p.v) of the integral in (11) is being evaluated. From (9) and (11), taking the Cauchy principal value,

$$P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j} \int_{-\infty}^{\infty} \frac{e^{\frac{\|\mathbf{m}\|^2}{2\sigma^2(1+2j\sigma^2 t)}}}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^n} dt. \tag{12}$$

[1]This form is rarely mentioned, but quite significant, because it tells us that the cumulative distribution function exists even if the integral is not absolutely convergent.

Expanding the exponential in the numerator of the integrand in (12) as a power series and interchanging the order of integration and summation, we obtain

$$P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j} \sum_{p=0}^{\infty} \frac{1}{p!}\left(\frac{\|\mathbf{m}\|^2}{2\sigma^2}\right)^p \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^{n+p}}. \tag{13}$$

If we let

$$I_n = \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^n}, \tag{14}$$

(13) can be written as

$$P(\Delta < 0) = \frac{1}{2} + \frac{e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{2\pi j} \sum_{p=0}^{\infty} \frac{1}{p!}\left(\frac{\|\mathbf{m}\|^2}{2\sigma^2}\right)^p I_{n+p} \tag{15}$$

To evaluate $P(\Delta < 0)$, we first need to compute the Cauchy principal value of the integral I_n and sum the ensuing infinite series in (15). We note that though I_n may not converge in the absolute sense, its Cauchy principle value exists, and is evaluated in the following section.

III. The Integral I_n

It is easy to verify that

$$\frac{1}{t(1+2j\sigma^2 t)^n} = \frac{1}{t} - 2j\sigma^2 \sum_{k=1}^{n} \frac{1}{(1+2j\sigma^2 t)^k}. \tag{16}$$

From (14) and (16), we now have

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(1-2jt)^{\frac{1}{2}}} \left[\frac{1}{t} - 2j\sigma^2 \sum_{k=1}^{n} \frac{1}{(1+2j\sigma^2 t)^k}\right] dt, \tag{17}$$

which, after changing the order of the integral and the summation can be written as

$$I_n = \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}} - 2j\sigma^2 \sum_{k=1}^{n} \int_{-\infty}^{\infty} \frac{dt}{(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^k}.$$

In the above, letting

$$J = \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}}, \tag{18}$$

$$J_k = 2j\sigma^2 \int_{-\infty}^{\infty} \frac{dt}{(1-2jt)^{\frac{1}{2}}(1+2j\sigma^2 t)^k}, \tag{19}$$

we can write (14) as

$$I_n = J - \sum_{k=1}^{n} J_k. \tag{20}$$

In the following subsections, we first show that J and J_k can be reduced to simple real and contour integrals respectively and then solve them.

A. The Real Integral

In (18), through a change of variables (from t to $-t$), we obtain

$$J = -\int_{-\infty}^{\infty} \frac{dt}{t(1+2jt)^{\frac{1}{2}}}. \quad (21)$$

Adding the expressions for J in (18) and (21),

$$2J = \int_{-\infty}^{\infty} \frac{dt}{t(1-2jt)^{\frac{1}{2}}} - \int_{-\infty}^{\infty} \frac{dt}{t(1+2jt)^{\frac{1}{2}}} \quad (22)$$

$$= \int_{-\infty}^{\infty} \frac{1}{t}\left[\frac{(1+2jt)^{\frac{1}{2}}-(1-2jt)^{\frac{1}{2}}}{(1+4t^2)^{\frac{1}{2}}}\right] dt. \quad (23)$$

Multiplying the numerator and denominator of the integrand in (23) by $\left[(1+2jt)^{\frac{1}{2}}+(1-2jt)^{\frac{1}{2}}\right]$,

$$J = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{t}\left[\frac{(1+2jt)-(1-2jt)}{(1+4t^2)^{\frac{1}{2}}\left\{(1+2jt)^{\frac{1}{2}}+(1-2jt)^{\frac{1}{2}}\right\}}\right] dt. \quad (24)$$

Canceling out all common factors,

$$J = 2j\int_{-\infty}^{\infty} \frac{dt}{(1+4t^2)^{\frac{1}{2}}\left\{(1+2jt)^{\frac{1}{2}}+(1-2jt)^{\frac{1}{2}}\right\}}. \quad (25)$$

In the above equation, we note that $(1-2j)^{\frac{1}{2}}$ is the complex conjugate of $(1+2j)^{\frac{1}{2}}$. Hence, the integrand in (25) is real as well as an even function of t. Thus, we get (see Appendix)

$$J = 4j\int_{0}^{\infty} \frac{dt}{(1+4t^2)^{\frac{1}{2}}\left\{(1+2jt)^{\frac{1}{2}}+(1-2jt)^{\frac{1}{2}}\right\}} \quad (26)$$

$$= j\pi. \quad (27)$$

B. The contour integral

The integral in (19) can be solved easily if it can be converted to a contour integral. Toward this end, we state the following Lemma [9].

Lemma 3.1: Let g(x) be a function of a real variable x such that $|g(x)|$ has a denominator different from zero for all real x and is of degree in excess of a unit higher than the degree of the numerator. Then

$$\int_{-\infty}^{\infty} g(x)dx = \int_C g(z)dz, \quad (28)$$

where C is a semicircle in the complex upper half-plane whose diameter is the real-axis and the integration is in the anti-clockwise sense.

For the integrand in (19), $k \geq 1$ and the degree of the denominator is greater than that of the numerator by $k + \frac{1}{2}$. From Lemma 3.1, we get

$$J_k = 2j\sigma^2 \int_C \frac{dz}{(1-2jz)^{\frac{1}{2}}(1+2j\sigma^2 z)^k} \quad (29)$$

$$= (2j\sigma^2)^{1-k}\int_C \frac{dz}{(1-2jz)^{\frac{1}{2}}(z-\frac{j}{2\sigma^2})^k}.$$

We now present a formula for finding the derivatives of an analytic function [9] and subsequently use it to evaluate I_k.

Lemma 3.2: If $g(z)$ is analytic in a domain D, then it has derivatives of all orders in D which are then also analytic functions in D. The value of the $(k-1)$th derivative at a point z_0 in D is given by the formula

$$g^{(k-1)}(z_0) = \frac{(k-1)!}{2\pi j}\int_L \frac{g(z)}{(z-z_0)^k} \quad (k=1,2,\ldots); \quad (30)$$

where L is any simple closed path in D which encloses z_0 and whose full interior belongs to D; the curve is traversed in the counterclockwise sense and $g^{(0)}(z_0) = g(z_0)$, by definition.

The function

$$f(z) = \frac{1}{(1-2jz)^{\frac{1}{2}}} \quad (31)$$

is analytic in the upper half-plane and C is a closed path in it. Since (29) can be written as

$$J_k = (2j\sigma^2)^{1-k}\int_C \frac{f(z)}{(z-\frac{j}{2\sigma^2})^k}dz, \quad (32)$$

and the point $\frac{j}{2\sigma^2}$ lies within C, using Lemma 3.2, we obtain

$$J_k = \frac{2\pi j(2j\sigma^2)^{1-k}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[\frac{1}{(1-2jz)^{\frac{1}{2}}}\right]_{z=\frac{j}{2\sigma^2}} \quad (33)$$

which, after some simplification, yields

$$J_k = \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}}, k=1, \quad (34)$$

$$= \frac{2\pi j\sigma}{(k-1)!}\frac{\frac{1}{2}\frac{3}{2}\cdots\frac{(2k-3)}{2}}{(1+\sigma^2)^{k-\frac{1}{2}}}, \quad 1 < k \leq n.$$

Substituting the expressions obtained in (27) and (34) in (20), we get

$$I_n = j\pi - \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}}\left[1+\sum_{k=2}^{n}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2k-3)}{2}}{(k-1)!(1+\sigma^2)^{k-1}}\right] \quad (35)$$

$$= j\pi - \frac{2\pi j\sigma}{\sqrt{1+\sigma^2}}\left[1+\sum_{k=1}^{n-1}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{(2k-1)}{2}}{k!(1+\sigma^2)^{k}}\right]$$

$$= 2\pi j\left[\frac{1}{2} - \frac{\sigma}{\sqrt{1+\sigma^2}}\sum_{k=0}^{n-1}\binom{2k}{k}\left\{\frac{1}{4(1+\sigma^2)}\right\}^k\right].$$

IV. Closed Form Expression for $E[Q(\|\mathbf{X}\|)]$

From (15) and (35),

$$P(\Delta < 0) = 1 - \frac{\sigma e^{-\frac{\|\mathbf{m}\|^2}{2\sigma^2}}}{\sqrt{1+\sigma^2}}\sum_{p=0}^{\infty}\sum_{k=0}^{n+p-1}\frac{1}{p!}\left(\frac{\|\mathbf{m}\|^2}{2\sigma^2}\right)^p \times \binom{2k}{k}\left\{\frac{1}{4(1+\sigma^2)}\right\}^k. \quad (36)$$

Let $\alpha = \frac{\|\mathbf{m}\|^2}{2\sigma^2}$ and $\beta = \frac{1}{1+\sigma^2}$. Then, changing the indices of summation,

$$P(\Delta < 0) = 1 - e^{-\alpha}\sqrt{1-\beta}(A+B) \quad (37)$$

where[2]

$$A = \sum_{p=0}^{\infty} \sum_{k=0}^{p} \binom{2k}{k} \left\{\frac{\beta}{4}\right\}^k \frac{\alpha^p}{p!}, \tag{38}$$

$$B = \sum_{p=0}^{\infty} \sum_{k=p}^{n+p-1} \binom{2k}{k} \left\{\frac{\beta}{4}\right\}^k \frac{\alpha^p}{p!}. \tag{39}$$

We define the *factorial function* [10] as

$$(\gamma)_q = \prod_{r=1}^{q} (\gamma + r - 1), \quad (\gamma)_0 = 1, \gamma \neq 0, \tag{40}$$

where q is a positive integer.

A. The B series

Since

$$\binom{2k}{k} = \frac{4^k \left(\frac{1}{2}\right)_k}{k!}, \tag{41}$$

we obtain

$$B = \sum_{p=0}^{\infty} \sum_{k=p}^{n+p-1} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \frac{\alpha^p}{p!}. \tag{42}$$

Changing the limits of summation in (42),

$$\begin{aligned} B &= \sum_{p=0}^{\infty} \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_{k+p} \beta^{k+p}}{(k+p)!} \frac{\alpha^p}{p!} \\ &= \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \sum_{p=0}^{\infty} \frac{\left(\frac{1}{2}+k\right)_p}{(k+1)_p} \frac{(\alpha\beta)^p}{p!} \\ &= \sum_{k=0}^{n-1} \binom{2k}{k} \left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2}+k; k+1; \alpha\beta\right), \end{aligned} \tag{43}$$

where ${}_1F_1(a;b;x)$ is the confluent hypergeometric function [10]. According to Kummer's formula for the confluent hypergeometric function,

$${}_1F_1(a;b;x) = e^x {}_1F_1(b-a;b;-x). \tag{44}$$

Using this result in (43), we obtain

$$B = \exp(\alpha\beta) \sum_{k=0}^{n-1} \binom{2k}{k} \left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2}; k+1; -\alpha\beta\right). \tag{45}$$

B. The A series

We rewrite (38) as

$$A = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \frac{\alpha^p}{p!} - \sum_{p=0}^{\infty} \sum_{k=p+1}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \frac{\alpha^p}{p!} \tag{46}$$

In the above,

$$\sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \frac{\alpha^p}{p!} = \left(\sum_{p=0}^{\infty} \frac{\alpha^p}{p!}\right) \left(\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!}\right). \tag{47}$$

[2]We assume that all the infinite series considered henceforth converge.

Since $|\beta| < 1$, the second sum on the right hand side of (47) is the binomial series, i.e.,

$$\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} = (1-\beta)^{-\frac{1}{2}}. \tag{48}$$

Thus,

$$A = \frac{e^\alpha}{\sqrt{1-\beta}} - S, \tag{49}$$

where

$$\begin{aligned} S &= \sum_{p=0}^{\infty} \sum_{k=p+1}^{\infty} \frac{\left(\frac{1}{2}\right)_k \beta^k}{k!} \frac{\alpha^p}{p!} \\ &= \sum_{p=0}^{\infty} \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+p} \beta^{k+p}}{(k+p)!} \frac{\alpha^p}{p!}. \end{aligned} \tag{50}$$

Following the steps in (43), (50) can be written as

$$S = \sum_{k=1}^{\infty} \binom{2k}{k} \left(\frac{\beta}{4}\right)^k {}_1F_1\left(\frac{1}{2}+k; k+1; \alpha\beta\right). \tag{51}$$

The above infinite series has a closed form expression [3]

$$S = \frac{2\exp(\frac{\alpha\beta}{2})}{\sqrt{1-\beta}} \tag{52}$$

$$\times \left[\exp\left\{\frac{\alpha}{2}(1+\beta)\right\} Q_1(u,w) - \frac{1}{2}(1+\sqrt{1-\beta}) I_0\left(\frac{\alpha\beta}{2}\right)\right], \tag{53}$$

where

$$\begin{aligned} u &= \sqrt{\frac{\alpha}{2}(2-\beta) - \frac{2}{\beta}\sqrt{1-\beta}} \\ w &= \sqrt{\frac{\alpha}{2}(2-\beta) + \frac{2}{\beta}\sqrt{1-\beta}}, \end{aligned} \tag{54}$$

and $Q_1(u,w)$ is the Marcum Q-function [6]. Substituting (52) in (49) gives us a closed form expression for A. Since we already have a compact expression for B in (45), replacing the infinite series for A and B in (36) by their respective closed form expressions, and noting from (2) and (4) that

$$E[Q(\|\mathbf{X}\|)] = \frac{1}{2} P(\Delta < 0), \tag{55}$$

we obtain an exact expression for $E[Q(\|\mathbf{X}\|)]$[3].

Corollary: When $\mathbf{X}$ is zero mean, from (35) and (36), we obtain

$$\begin{aligned} P(\Delta < 0) &= \frac{1}{2} + \frac{I_n}{2\pi j} \\ &= 1 - \frac{\sigma}{\sqrt{1+\sigma^2}} \sum_{k=0}^{n-1} \binom{2k}{k} \left\{\frac{1}{4(1+\sigma^2)}\right\}^k. \end{aligned} \tag{56}$$

Substituting the above in (55) leads to the well known result [4]

$$E[Q(\|\mathbf{X}\|)] = \frac{1}{2}\left[1 - \frac{\sigma}{\sqrt{1+\sigma^2}} \sum_{k=0}^{n-1} \binom{2k}{k} \left\{\frac{1}{4(1+\sigma^2)}\right\}^k\right] \tag{57}$$

[3]Lindsey, in [3], using a fundamentally different approach, has obtained a closed form expression in a completely different form.

V. Example: Average Probability of Error for Nakagami-m fading channels

If a random variable α is Nakagami-m distributed, the random variable $\gamma = \frac{\alpha^2 \varepsilon_b}{N_0}$ has the probability density function [6]

$$p(\gamma) = \frac{m^m}{\Gamma(m)\bar{\gamma}} \gamma^{m-1} e^{-m\gamma/\bar{\gamma}}, \tag{58}$$

where $\bar{\gamma} = \frac{E(\alpha^2)\varepsilon_b}{N_0}$. For fading channels, the average probability of error is given by

$$P_e = \int_0^\infty Q(a\sqrt{\gamma}) p_\gamma(\gamma) d\gamma, \tag{59}$$

where a is a constant that depends on the specific modulation/detection combination [1]. We note that γ has the same distribution as R (when $\mathbf{X}$ has zero mean) with $\sigma^2 = \frac{\bar{\gamma}}{2m}$ when m is an integer [6]. Thus, after accounting for the constant a in (59), the average probability of error for a Nakagami-m fading channel is obtained as

$$P_e = \frac{1}{2}\left[1 - \sqrt{\frac{a^2\bar{\gamma}}{2m + a^2\bar{\gamma}}} \sum_{k=0}^{m-1} \binom{2k}{k} \left\{\frac{2m}{4(2m + a^2\bar{\gamma})}\right\}^k\right] \tag{60}$$

by substituting $\sigma^2 = \frac{a^2\bar{\gamma}}{2m}$ and replacing n by m in (57). We note that exactly the same result has been arrived at using a different approach in [1], equation (5.18).

Appendix

Let $1 + 2jt = re^{j\theta}$, where $r = (1 + 4t^2)^{\frac{1}{2}}$ and $\cos\theta = \frac{1}{r}$. Since

$$(1 + 2jt)^{\frac{1}{2}} + (1 - 2jt)^{\frac{1}{2}} = 2r^{\frac{1}{2}} \cos\frac{\theta}{2}, \tag{61}$$

and

$$\begin{aligned} \cos\frac{\theta}{2} &= \sqrt{\frac{1}{2}(1 + \cos\theta)} \\ &= \sqrt{\frac{1}{2}\left(1 + \frac{1}{r}\right)}, \end{aligned} \tag{62}$$

the integrand in (26)

$$\frac{1}{(1 + 4t^2)^{\frac{1}{2}} \left\{(1 + 2jt)^{\frac{1}{2}} + (1 - 2jt)^{\frac{1}{2}}\right\}} = \frac{1}{2r\sqrt{\frac{1+r}{2}}}. \tag{63}$$

Hence,

$$\begin{aligned} J &= 4j \int_0^\infty \frac{dt}{(1 + 4t^2)^{\frac{1}{2}} \left\{(1 + 2jt)^{\frac{1}{2}} + (1 - 2jt)^{\frac{1}{2}}\right\}} \\ &= 2\sqrt{2}j \int_0^\infty \frac{dt}{(1 + 4t^2)^{\frac{1}{2}} \left\{1 + (1 + 4t^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}}}. \end{aligned} \tag{64}$$

Substituting $t = \frac{\tan\phi}{2}$ in the above,

$$J = \sqrt{2}j \int_0^{\frac{\pi}{2}} \frac{\sec\phi}{\sqrt{1 + \sec\phi}} d\phi. \tag{65}$$

Let $\sqrt{1 + \sec\phi} = \sqrt{2}\sec\psi$. Then

$$\frac{\sec\phi}{\sqrt{1 + \sec\phi}} d\phi = \sqrt{2} d\psi. \tag{66}$$

The integral in (65) now becomes

$$\begin{aligned} J &= 2j \int_0^{\frac{\pi}{2}} d\psi \\ &= j\pi. \end{aligned} \tag{67}$$

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