Exact Error Analysis for Decode and Forward Cooperation with Maximal Ratio Combining

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***Abstract*—In this paper, we provide exact expressions for the bit error rate (BER) for single relay maximal ratio combining (MRC) based decode and forward (DF) cooperative systems in Nakagami-m fading. This is done by expressing the decision variable as a sum of gamma conditionally Gaussian (CG) random variables. The characteristic function (CF) of gamma CG variables is then derived and used to obtain the BER expressions using the Gil-Pelaez inversion formula. A tight closed form approximation for the BER is also derived and used to obtain the diversity order. Numerical results, including simulations, are provided to verify the validity of the derived analytical expressions.**

***Index Terms*—DF Cooperative diversity, gamma conditionally Gaussian, MRC.**

I. Introduction

The decode and forward (DF) protocol [1] for cooperative communications has been a subject of considerable interest to researchers in recent years. Some of the popular DF receivers in the literature are the maximum likelihood (ML) receiver for DF proposed in [2], the adaptive maximal ratio combining (MRC) receiver in [3] and the cooperative MRC (C-MRC) receiver in [4]. All the above receivers assume some knowledge of the source (S) - relay (R) link in order to achieve diversity gain. Approximate expressions for the bit error rate (BER) were available in [2], [3] while only diversity analysis was carried out in [4]. Diversity analysis was also carried out in [2] for ML-DF cooperation while corresponding analysis for simple MRC-DF was done in [5].

While simple MRC does not provide any significant diversity gain, its application in adaptive MRC results in full diversity[1]. Thus, exact BER analysis for MRC-DF is of interest and the focus of this work. The decision variable for MRC-DF is first shown to be the sum of two gamma conditionally Gaussian variables [6]. Conditionally Gaussian (CG) distributions had been proposed in [6] to obtain expressions for the BER for ML-DF based cooperation. While these distributions are widely encountered in fading channels, to the best of our knowledge, a framework for their application was first proposed only in [7]. In this paper, following the approach in [7], we obtain an expression for the characteristic function (CF) of a gamma conditionally Gaussian variable. Using the Gil-Pelaez inversion formula [8], an exact expression for the BER for MRC-DF is then obtained. Further, approximate expressions for the BER are obtained and shown to be simpler than those in [3]. As a corollary, the diversity order for MRC-DF is also obtained from the approximate analysis. The CF based approach in this paper, is not only simpler, as will be evident from the following analysis, but also results in expressions that lend useful insights into the variation of the BER performance with various system parameters.

[1]This has been claimed in [3] based on numerical results.

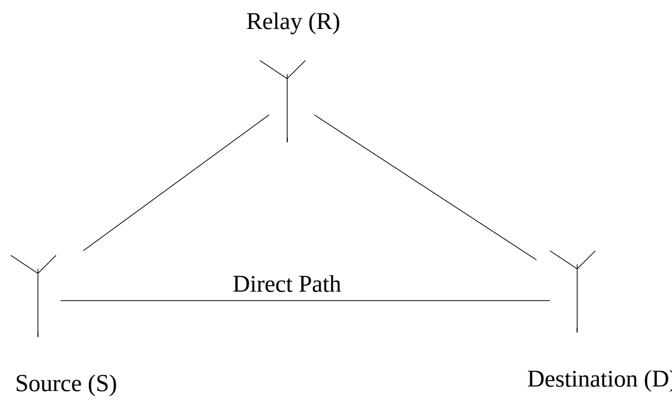

Fig. 1. Three node cooperative diversity system.

II. MRC-DF Cooperation

For the cooperative system in Figure 1, without loss of generality, let h represent the Nakagami-m channel gain with fading figures m and Ω, E the transmit power at a node, x the transmitted symbol at a node, and the subscripts s and r the source and relay parameters respectively. The decision criterion for binary phase shift keying (BPSK) based MRC may be obtained from [3] as

$$X + Y \underset{-1}{\overset{1}{\gtrless}} 0, \tag{1}$$

where $X \sim \mathcal{N}\left(a_s x_s |h_s|^2, b_s |h_s|^2\right)$ and $Y \sim \mathcal{N}\left(a_r x_r |h_r|^2, b_r |h_r|^2\right)$ for $|h_i|^2 \sim \mathcal{G}\left(c_i, m_i\right)$ being gamma distributed with scale parameter c_i and order m_i [9], and $a_i = \frac{4E_i}{N_0}, b_i = \frac{8E_i}{N_0}, c_i = \frac{m_i}{\Omega_i}, i \in \{s, r\}$. It is obvious that both X and Y are gamma CG [6]. Assuming equal probability of the symbol $x_s = \{1, -1\}$ transmitted at the source, the average probability of error for the MRC-DF cooperative diversity system can be expressed as

$$P_e = \epsilon P\left(X+Y<0|x_s=1,x_r=-1\right) + (1-\epsilon)P\left(X+Y<0|x_s=1,x_r=1\right). \quad (2)$$

where ϵ is the BER on the S-R link [10, (5.18)].

III. BER ANALYSIS

A. Exact Analysis

The characteristic function (CF) of X can be expressed as (Appendix A)

$$\Phi_X(\jmath\omega) = \frac{1}{\left(1-\jmath x_s\frac{4\bar{\gamma}_s}{m_s}\omega+\frac{4\bar{\gamma}_s}{m_s}\omega^2\right)^{m_s}}, \quad (3)$$

where $\bar{\gamma}_i = \frac{\Omega_i E_i}{N_0}$. Since X and Y are independent,

$$\Phi_{X+Y}(\jmath\omega) = \frac{1}{\prod_{j=1,2}\left(1-\jmath\omega\alpha_j\right)^{m_s}\left(1-\jmath\omega\beta_j\right)^{m_r}}, \quad (4)$$

where

$$\alpha_1,\alpha_2 = 2\sqrt{\frac{\bar{\gamma}_s}{m_s}}\left(x_s\sqrt{\frac{\bar{\gamma}_i}{m_s}}\pm\sqrt{1+\frac{\bar{\gamma}_i}{m_s}}\right), \quad (5)$$

$$\beta_1,\beta_2 = 2\sqrt{\frac{\bar{\gamma}_r}{m_r}}\left(x_r\sqrt{\frac{\bar{\gamma}_r}{m_r}}\pm\sqrt{1+\frac{\bar{\gamma}_r}{m_r}}\right). \quad (6)$$

Thus, noting that $\alpha_1,\beta_1>0$ and $\alpha_2,\beta_2<0$, from [11],

$$P(X+Y<0 \mid x_s=1,x_r) = -\sum \operatorname*{Res}_{z=\frac{1}{\alpha_2},\frac{1}{\beta_2}} \frac{\Phi_{X+Y}(z)}{z}. \quad (7)$$

Since

$$\operatorname*{Res}_{z=\frac{1}{\alpha_2}} \frac{\Phi_{X+Y}(z)}{z} = \frac{(-\alpha_2)^{-m_s}}{(m_s-1)!} \times \frac{d^{m_s-1}}{dz^{m_s-1}}\left[\frac{1}{z\left(1-z\alpha_1\right)^{m_s}} \times \frac{1}{\prod_{j=1,2}\left(1-z\beta_j\right)^{m_r}}\right]_{z=\frac{1}{\alpha_2}}, \quad (8)$$

from Appendix B, the residue in (8) can be expressed as (9). The residue at $z=\frac{1}{\beta_2}$ can be trivially evaluated using (9) leading to a closed form expression for the conditional BER in (7), which results in an exact expression for the BER in (2).

B. Approximate Analysis

From (3) and [11, (75-76)]

$$\begin{aligned} P(X+Y<0|x_s=1,x_r=1) &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\sin^2\theta}{\frac{\bar{\gamma}_s}{m_s}+\sin^2\theta}\right)^{m_s}\left(\frac{\sin^2\theta}{\frac{\bar{\gamma}_r}{m_r}+\sin^2\theta}\right)^{m_r} d\theta \\ &\le \left(\frac{m_s}{\bar{\gamma}_s}\right)^{m_s}\left(\frac{m_r}{\bar{\gamma}_s}\right)^{m_r}\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(\sin^2\theta\right)^{m_s+m_r}d\theta \quad (10) \\ &= \frac{1}{2\sqrt{\pi}}\left(\frac{m_s}{\bar{\gamma}_s}\right)^{m_s}\left(\frac{m_r}{\bar{\gamma}_s}\right)^{m_r}\frac{\Gamma\left(m_s+m_r+\frac{1}{2}\right)}{\Gamma\left(m_s+m_r+1\right)} \quad (11) \end{aligned}$$

from [12], with $\bar{\gamma}_i = \frac{\Omega_i E_i}{N_0}, i\in\{r,s\}$. The above expression is a relatively loose but simple bound that is likely to become tighter at high signal to noise ratio (SNR). Several tighter bounds have been discussed in [11] and can be used in place of (11). Note that the above technique cannot be used for $P\left(X+Y<0|x_s=1,x_r=-1\right)$. To evaluate this probability, we use the approximation in [3] to obtain, for $\gamma_i = \frac{E_i}{N_0}|h_i|^2, i\in\{r,s\}$ [11],

$$\begin{aligned} P(X+Y<0|x_s=1,x_r=-1) &\approx P\left(\gamma_s<\gamma_r\right) \\ &= -\operatorname*{Res}_{z=-\frac{m_r}{\bar{\gamma}_r}}\left\{\frac{1}{z(1-\frac{\bar{\gamma}_s}{m_s}z)^{m_s}(1+\frac{\bar{\gamma}_r}{m_r}z)^{m_r}}\right\} \\ &= -\frac{1}{(m_r-1)!}\left(\frac{m_r}{\bar{\gamma}_r}\right)^{m_r} \times \frac{d^{m_r-1}}{dz^{m_r-1}}\left[\frac{1}{z(1-\frac{\bar{\gamma}_s}{m_s}z)^{m_s}}\right]_{z=-\frac{m_r}{\bar{\gamma}_r}}, \end{aligned} \quad (12)$$

which can be expressed as (Appendix C)

$$P\left(X+Y<0|x_s=1,x_r=-1\right) = \left[1+\left(\frac{m_r}{\bar{\gamma}_r}\right)\left(\frac{\bar{\gamma}_s}{m_s}\right)\right]^{-m_s} \times \sum_{k=0}^{m_r-1}\binom{m_s+k-1}{k} \times \left[\frac{\left(\frac{m_r}{\bar{\gamma}_r}\right)\left(\frac{\bar{\gamma}_s}{m_s}\right)}{1+\left(\frac{m_r}{\bar{\gamma}_r}\right)\left(\frac{\bar{\gamma}_s}{m_s}\right)}\right]^k. \quad (13)$$

C. Diversity Order

From [13, 14-4-18],

$$\epsilon \approx \binom{2m-1}{m}\left(\frac{1}{4\bar{\gamma}}\right)^m \quad \text{(high SNR)}, \quad (14)$$

$$\begin{aligned} \operatorname*{Res}_{z=\frac{1}{\alpha_2}}\frac{\Phi_{X+Y}(z)}{z} = &-\frac{\alpha_1^{-m_s}(\beta_1\beta_2)^{-m_r}}{[\Gamma(m_s)\Gamma(m_r)]^2}\sum_{k=0}^{m_s-1}\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}\binom{m_s-1}{k}\binom{k}{k_1}\binom{k_1}{k_2}\Gamma(m_s-k)\Gamma(m_s+k-k_1)\Gamma(m_r+k_2) \\ &\times\Gamma(m_r+k_1-k_2)(-\alpha_2)^{-k}(\alpha_1^{-1}-\alpha_2^{-1})^{-m_s-k+k_1}(\beta_1{}^{-1}-\alpha_2^{-1})^{-m_r-k_2}(\beta_2{}^{-1}-\alpha_2^{-1})^{-m_r-k_1+k_2}. \end{aligned} \quad (9)$$

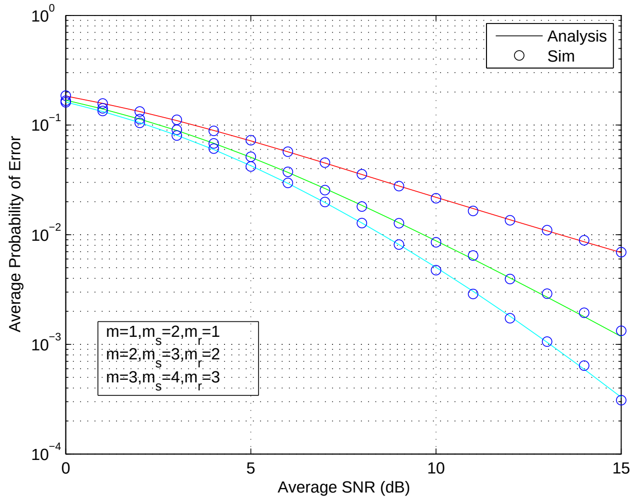

Fig. 2. Comparison of analytical and simulation results. Sequence of plots in the same order as the Nakagami severities listed in the box.

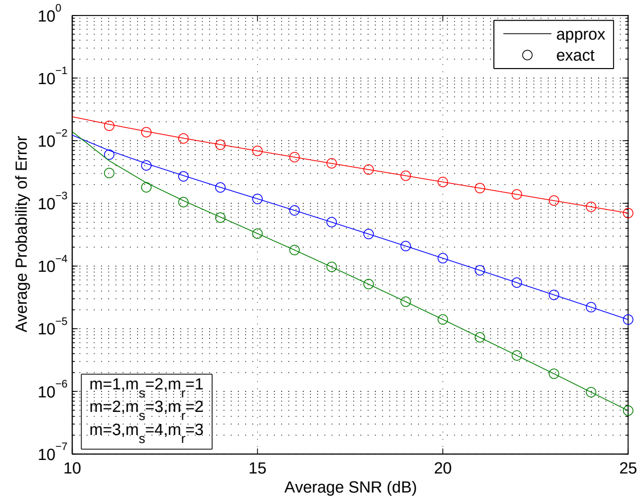

Fig. 3. High SNR comparison of exact and approximate BER expressions. Sequence of plots in the same order as the Nakagami severities listed in the box.

where m and $\bar{\gamma}$ are the fading parameters on the S-R link. Since $\epsilon \ll 1$ for high SNR, from (2),

$$P_e \leq \epsilon P\left(X+Y<0|x_s=1,x_r=-1\right) + P\left(X+Y<0|x_s=1,x_r=1\right). \quad (15)$$

Since $\left[\frac{\left(\frac{m_r}{\bar{\gamma}_r}\right)\left(\frac{\bar{\gamma}_s}{m_s}\right)}{1+\left(\frac{m_r}{\bar{\gamma}_r}\right)\left(\frac{\bar{\gamma}_s}{m_s}\right)}\right] < 1$, using the identity

$$\sum_{k=0}^{q}\binom{p+k}{k} = \binom{p+q+1}{p+1}, \quad p,q,k \in \mathbb{Z}, \quad (16)$$

in (13),

$$P\left(X+Y<0|x_s=1,x_r=-1\right) \leq \binom{m_s+m_r-1}{m_s} \times \left[1+\left(\frac{m_r}{\bar{\gamma}_r}\right)\left(\frac{\bar{\gamma}_s}{m_s}\right)\right]^{-m_s}. \quad (17)$$

From (11),(14),(15) and (17), it is obvious that the diversity order for MRC based DF cooperation is[2]

$$d = \min(m, m_s + m_r). \quad (18)$$

IV. Results and Discussion

In the following, the respective Nakagami severities and average SNRs on the S-R, S-D and R-D links are denoted by $(m,\bar{\gamma})$, $(m_s,\bar{\gamma}_s)$ and $(m_r,\bar{\gamma}_r)$. Further, we define $\xi = \frac{\bar{\gamma}_r}{\bar{\gamma}_s}$. A comparison of the simulation and analytical results is shown in Figure 2 for different Nakagami severities. As we can see, the simulation results closely follow the analytical ones, validating the BER expressions obtained in section III-A. For higher values of m, m_s and m_r, the BER keeps improving, which is expected. We have assumed ξ to be unity, for the above plots.

[2]This result has been obtained in [5] through a different approach.

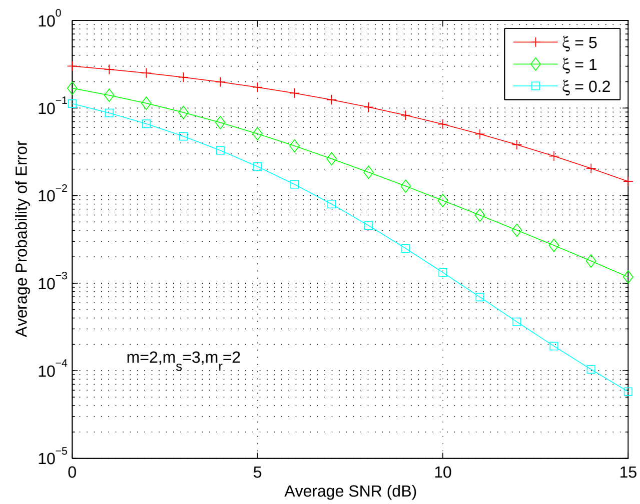

Fig. 4. Variation of the BER with m and ξ. System performance improves when $\xi < 1$.

In the high SNR region (10-25 dB), the closeness of the BER approximations in section III-B is investigated by plotting the BER for a system with different Nakagami severities in Figure 3. The BER curves obtained through approximate analysis are quite close to those obtained through exact analysis. As can be seen from (11) and (13), approximate expressions for the BER involve very simple functions and are hence practically useful. Thus, for systems operating in the high SNR range, approximate BER expressions can be used in place of the exact ones. A simple application could be the adaptive relaying scheme discussed in [3]. In Figure 4, for

$m = 2, m_s = 3$, and $m_r = 2$, $\xi = 0.2, 1$ or 5. For $\xi < 1$, i.e. when the relay power is lesser than the source, we find that the BER improves significantly. This happens to strong S-R and S-D links. On the other hand, for $\xi > 1$, system performance is relatively poor. This is because the errors on the S-R link increase due to reduced source power and are then propagated on the R-D link due to higher relay power. Thus, for $\xi < 1$, there is an obvious improvement in system performance. This is evident from (17), clearly emphasizing the utility of (13). An alternative expression for (13) was obtained in in [3, (7)], but does not provide the above insights.

V. Conclusions

In this paper, we have obtained closed form expressions for the BER for MRC based DF cooperative systems through a CF based approach. Practically useful approximations for the BER were also obtained and used to derive the diversity order. Numerical results show that the simulations closely follow the analysis. Also, by plotting the BER for high SNR, the approximations were shown to be close to the exact results. The effect of the link SNR imbalance on the BER could be clearly observed from the analytical expressions and confirmed through graphical plots. The extension of the results obtained in this work for multibranch cooperation is the focus of future research.

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Appendix A

The CF of $|h_s|^2$ can be expressed as [10]

$$\Phi_{|h_s|^2}(\jmath\omega) = \frac{1}{\left(1 - \jmath\frac{\omega}{c_s}\right)^{m_s}}. \tag{19}$$

Invoking its *Gaussian* property, the CF of $X \mid |h_s|^2$ is given by

$$\Phi_{X||h_s|^2}(\jmath\omega) = \exp\left\{\left(\jmath\omega a_s - \frac{b_s}{2}\omega^2\right)|h_s|^2\right\}. \tag{20}$$

Averaging the above over $|h_s|^2$, we have [7]

$$\begin{aligned}\Phi_X(\jmath\omega) &= \int_{-\infty}^{\infty} \exp\left\{\jmath\left(\omega a_s + \jmath\frac{b_s}{2}\omega^2\right)y\right\} p_{|h_s|^2}(y)dy \\ &= \Phi_A\left\{\jmath\left(a\omega + \jmath\frac{b}{2}\omega^2\right)\right\},\end{aligned} \tag{21}$$

which, from (19), can be expressed as (3).

Appendix B

Repeatedly applying the Leibniz rule for differentiation,

$$\begin{aligned}&\frac{d^{m_s-1}}{dz^{m_s-1}}\left[\frac{1}{z\left(1-z\alpha_1\right)^{m_s}\prod_{j=1,2}\left(1-z\beta_j\right)^{m_r}}\right] \\ &= \sum_{k=0}^{m_s-1}\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}\binom{m_s-1}{k}\binom{k}{k_1}\binom{k_1}{k_2} \\ &\times \frac{d^{m_s-1-k}(z^{-1})}{dz^{m_s-1-k}}\frac{d^{k_2}}{dz^{k_2}}\left[\frac{1}{\left(1-z\beta_1\right)^{m_r}}\right] \\ &\times \frac{d^{k-k_1}}{dz^{k-k_1}}\left[\frac{1}{\left(1-z\alpha_1\right)^{m_s}}\right] \\ &\times \frac{d^{k_1-k_2}}{dz^{k_1-k_2}}\left[\frac{1}{\left(1-z\beta_2\right)^{m_r}}\right],\end{aligned} \tag{22}$$

which can be expressed as

$$\begin{aligned}&\frac{d^{m_s-1}}{dz^{m_s-1}}\left[\frac{1}{z\left(1-z\alpha_1\right)^{m_s}\prod_{j=1,2}\left(1-z\beta_j\right)^{m_r}}\right] \\ &= -\sum_{k=0}^{m_s-1}\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}(-1)^{m_s-k}\Gamma(m_s-k)z^{k-m_s} \\ &\times \alpha_1^{k-k_1}\frac{\Gamma(m_s+k-k_1)}{\Gamma(m_s)}(1-\alpha_1 z)^{-m_s-k+k_1} \\ &\times \beta_1^{k_2}\frac{\Gamma(m_r+k_2)}{\Gamma(m_r)}(1-\beta_1 z)^{-m_r-k_2}\beta_2^{k_1-k_2} \\ &\times \frac{\Gamma(m_r+k_1-k_2)}{\Gamma(m_r)}(1-\beta_2 z)^{-m_r-k_1+k_2},\end{aligned} \tag{23}$$

since

$$\frac{d^{m_s-1-k}\left(z^{-1}\right)}{dz^{m_s-1-k}} = (-1)^{m_s-1-k}\Gamma(m_s-k)z^{k-m_s}, \tag{24}$$

$$\frac{d^k}{dz^k}\left[\frac{1}{(1-\alpha_1 z)^{m_s}}\right] = \alpha_1^k\frac{\Gamma(m_s+k)}{\Gamma(m_s)}(1-\alpha_1 z)^{-m_s-k}, \tag{25}$$

(25) being used to evaluate the remaining two derivatives. Substituting $z = \frac{1}{\alpha_2}$ in (23), from (8) we obtain (9).

APPENDIX C

From (24) and (25), (12) can be expressed using the Leibniz rule as

$$
\begin{aligned}
&P\left(X+Y<0|x_s=1,x_r=-1\right)= \\
&\quad\frac{1}{(m_r-1)!}\left(\frac{m_r}{\bar{\gamma}_r}\right)^{m_r}\sum_{k=0}^{m_r-1}\binom{m_r-1}{k}(-1)^{m_r-k} \\
&\qquad\times\frac{(m_r-1-k)!(m_s+k-1)!}{(m_s-1)!} \\
&\qquad\times\left(-\frac{m_r}{\bar{\gamma}_r}\right)^{-(m_r-k)}\left(\frac{\bar{\gamma}_s}{m_s}\right)^{k} \\
&\qquad\times\left[1+\left(\frac{m_r}{\bar{\gamma}_r}\right)\left(\frac{\bar{\gamma}_s}{m_s}\right)\right]^{-(m_s+k)},
\end{aligned}
\tag{26}
$$

which, after some algebra, results in

$$
\begin{aligned}
&P\left(X+Y<0|x_s=1,x_r=-1\right) \\
&=\frac{1}{(m_r-1)!}\sum_{k=0}^{m_r-1}\binom{m_r-1}{k}\frac{(m_r-1-k)!(m_s+k-1)!}{(m_s-1)!} \\
&\times\left(\frac{m_r}{\bar{\gamma}_r}\right)^{k}\left(\frac{\bar{\gamma}_s}{m_s}\right)^{k}\left[1+\left(\frac{m_r}{\bar{\gamma}_r}\right)\left(\frac{\bar{\gamma}_s}{m_s}\right)\right]^{-(m_s+k)},
\end{aligned}
\tag{27}
$$

yielding (13) upon further simplification.