Conditionally Gaussian Distributions and their Application in the Performance of Maximum Likelihood Decode and Forward Cooperative Systems

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***Abstract*—The statistics of gamma conditionally Gaussian (CG) random variables are derived in closed form. These variables can be loosely defined to be normally distributed with mean and variance proportional to a gamma random variable. In this paper, we provide exact expressions for the bit error rate (BER) for single relay maximum likelihood (ML) decode and forward (DF) cooperative systems in Nakagami-m fading for binary phase shift keying (BPSK). This is done by expressing the ML decision variable in terms of functions of gamma CG random variables. For the piecewise linear (PL) approximation to the ML detector, a closed form expression for the BER is obtained. Simulation results are provided to verify the validity of the derived analytical expressions.**

***Index Terms*—Cooperative diversity, gamma conditionally Gaussian, ML, Nakagami-m fading.**

I. Introduction

The BER analysis for amplify and forward (AF) based cooperation is well researched with the availability of closed form expressions based on the statistics of the harmonic mean of two independent random variables [1], [2]. However, very few results for the error analysis for decode and forward (DF) cooperation are available in the literature [3]. In particular, the performance of the optimum receiver for DF cooperation is important as it provides a benchmark for simpler DF based techniques. While the maximum likelihood (ML) detector and its practical alternative, the piecewise linear (PL) receiver for DF, were proposed in [4], their bit error rate (BER) performance was not seriously investigated until now. Expressions for the BER for ML-DF cooperative systems have been obtained in [5] for Rayleigh fading.

Conditionally Gaussian (CG) distributions were first defined in [5] and used to obtain expressions for the BER for ML-DF based cooperation, though a framework for their application was already available in [6]. The decision variable for M-ary phase shift keying (MPSK) is CG, for which the characteristic function (CF), was derived in [6] to evaluate the symbol error probability (SEP). In this paper, following the approach in [5], we obtain closed form expressions for the the statistics of gamma CG random variables. For binary phase shift keying (BPSK) being used at the nodes for transmission, these statistics are used to obtain exact and closed form expressions for the BER for ML and PL-DF cooperation [4] in Nakagami-m fading for integer m.

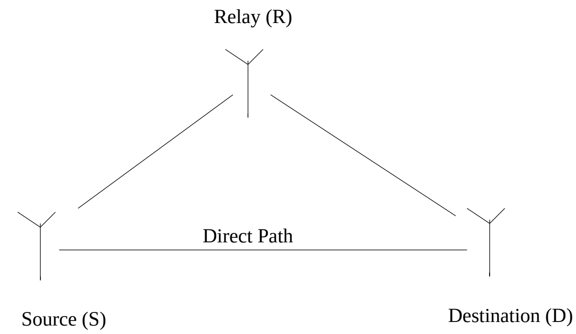

Fig. 1. Three node cooperative diversity system.

II. Conditionally Gaussian Distributions

A. Preliminaries

Definition 2.1: Z is gamma CG with parameters $a, b > 0$ if $Z \mid A \sim \mathcal{N}(aA, bA), A \sim \mathcal{G}(c, m)$ being Gamma distributed [7] with scale parameter $c > 0$ and order $m > 0$.

Definition 2.2: For $0 < \nu < 1$,

$$f(t) = \ln \frac{\nu + e^t}{1 + \nu e^t}, \quad g(t) = \frac{e^{-t} - \nu}{1 - \nu e^{-t}}. \tag{1}$$

Proposition 2.1: (*Exponential approximation*) $g(t) \approx e^{-t}$, $t > 0$, $\nu \ll 1$.

Proof: From (1), for $0 < \nu < 1$

$$|g(t) - e^{-t}| = |-\nu| \frac{|1 - e^{-2t}|}{|1 - \nu e^{-t}|} < \frac{\nu}{1 - \nu e^{-t}}, \quad t > 0, \tag{2}$$

resulting in the desired approximation. Henceforth, symbols f and g denote the functions in (1).

Lemma 2.1: For any constants $a, b > 0, c > 0$ and $m \in \mathbb{Z}^+$, the integral defined by

$$\mathcal{I}_m(a, b, c) = \frac{c^m}{(m-1)!} \int_0^\infty x^{m-1} Q\left(\frac{ax + b}{\sqrt{x}}\right) e^{-cx} dx \tag{3}$$

can be expressed as

$$\mathcal{I}_m(a,b,c)=\frac{c^m e^{-\kappa b}}{\kappa\mu^{2m-1}}\sum_{\mathbf{l}.\mathbf{v}_{m-1}=m-1}\sum_{\mathbf{q}.\mathbf{v}_n=n}\mu^n n!\times\prod_{j=1}^{n}\left[b\delta_{j-1}+\frac{1}{\kappa^j}+\frac{1}{\mu^j}\right]^{q_j}\frac{1}{j^{q_j}q_j!}\times\prod_{k=1}^{m-1}\left[\frac{(2k-3)!!}{k!}\right]^{l_k}\frac{1}{l_k!},\quad(4)$$

where $Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt, \kappa=a+\sqrt{a^2+2c}, \mu=\kappa-a, \delta_{(\cdot)}$ is the Kronecker delta function, $\mathbf{l}=(l_1,l_2,\ldots,l_{m-1}), \mathbf{q}=(q_1,q_2,\ldots,q_n), 0\le l_k\le m-1, 0\le q_j\le n, \mathbf{v}_k=(1,2,\ldots,k)$, $(\cdot)$ denoting the inner product and $n=\sum_{k=1}^{m-1}l_k$.

Corollary: The partial derivative of $\mathcal{I}_m$ with respect to the variable b is

$$\mathcal{J}_m(a,b,c)=\frac{c^m e^{-\kappa b}}{\mu^{2m-1}}\sum_{\mathbf{l}.\mathbf{v}_{m-1}=m-1}\sum_{\mathbf{q}.\mathbf{v}_n=n}\frac{\mu^n n!}{q_1!}\times\prod_{k=1}^{m-1}\left[\frac{(2k-3)!!}{k!}\right]^{l_k}\frac{1}{l_k!}\prod_{j=2}^{n}\left[\frac{1}{\kappa^j}+\frac{1}{\mu^j}\right]^{q_j}\frac{1}{j^{q_j}q_j!}\times\left\{(1-\delta_{q_1})\left[\frac{q_1-1}{\kappa}-\left(b+\frac{1}{\mu}\right)\right]\times\left(b+\frac{1}{\kappa}+\frac{1}{\mu}\right)^{q_1-1}-\delta_{q_1}\right\}.\quad(5)$$

Proof: See Appendix A.

B. Closed form expressions for the statistical functions of gamma CG and related distributions

Theorem 2.1: The Gamma CG variable Z in Definition 2.1 with parameters a,b,c and m, has the cumulative distribution function (CDF) and probability density function (PDF)

$$F_Z(z)=\begin{cases}1-\mathcal{I}_m\left(-\frac{a}{\sqrt{b}},\frac{z}{\sqrt{b}},c\right) & z\ge 0\\ \mathcal{I}_m\left(\frac{a}{\sqrt{b}},-\frac{z}{\sqrt{b}},c\right) & z<0\end{cases},$$
$$p_Z(z)=\begin{cases}-\frac{1}{\sqrt{b}}\mathcal{J}_m\left(-\frac{a}{\sqrt{b}},\frac{z}{\sqrt{b}},c\right) & z\ge 0\\ -\frac{1}{\sqrt{b}}\mathcal{J}_m\left(\frac{a}{\sqrt{b}},-\frac{z}{\sqrt{b}},c\right) & z<0\end{cases}\quad(6)$$

for $\mathcal{I}_m(.,.,.),\mathcal{J}_m(.,.,.)$ defined in (4) and (5) respectively.

Corollary 1: The CDF of $V=f(Z)$ can be expressed as

$$F_V(z)=\begin{cases}1 & z>\ln\frac{1}{\nu}\\ 1-\mathcal{I}_m\left(-\frac{a}{\sqrt{b}},\frac{-\ln g(z)}{\sqrt{b}},c\right) & 0<z<\ln\frac{1}{\nu}\\ \mathcal{I}_m\left(\frac{a}{\sqrt{b}},\frac{\ln g(z)}{\sqrt{b}},c\right) & \ln\nu<z<0\\ 0 & z<\ln\nu\end{cases}.\quad(7)$$

Corollary 2: The Nth moment of Z can be expressed as

$$E[Z^N]=\frac{Nc^m}{b^m\mu_Z^{2m-1}}\times\sum_{\upsilon\in\{1,-1\}}\frac{\upsilon^N}{\kappa_Z}\sum_{\mathbf{l}.\mathbf{v}_{m-1}=m-1}\prod_{k=1}^{m-1}\left[\frac{(2k-3)!!}{k!}\right]^{l_k}\frac{1}{l_k!}\times\sum_{\mathbf{q}.\mathbf{v}_n=n}\frac{\mu_Z^n n!}{q_1!}\prod_{j=2}^{n}\left[\frac{1}{\kappa_Z^j}+\frac{1}{\mu_Z^j}\right]^{q_j}\frac{1}{j^{q_j}q_j!}\times\sum_{\rho=0}^{q_1}\binom{q_1}{\rho}\kappa_Z{}^{-N-\rho}\left(\frac{1}{\kappa_Z}+\frac{1}{\mu_Z}\right)^{q_1-\rho}\times\Gamma(N+\rho),\quad(8)$$

where $\kappa_Z=\frac{\sqrt{a^2+2bc}-\upsilon a}{b}, \quad \mu_Z=\frac{\sqrt{a^2+2bc}}{b}$.

Proof: See Appendix B.

III. ML-DF COOPERATION

For the classic three node cooperative system in Figure 1, without loss of generality, assuming h to represent the Nakagami-m channel gain with fading figures m and Ω, E the transmit power at a node, x the transmitted symbol at a node, and the subscripts s and r the source and relay parameters respectively, the ML decision criterion at the destination for binary phase shift keying (BPSK) modulation may be obtained from [4], [5] as $X+f(Y)\underset{-1}{\overset{1}{\gtrless}}0$, where $X\sim\mathcal{N}\left(a_s h_s^2, b_s h_s^2\right)$, $Y\sim\mathcal{N}\left(a_r h_r^2, b_r h_r^2\right)$ are gamma CG with $h_i^2\sim\mathcal{G}(c_i,m_i), a_i=\frac{4E_i x_i}{N_0}, b_i=\frac{8E_i}{N_0}, c_i=\frac{m_i}{\Omega_i}, i\in\{s,r\}$. Also, $f(\cdot)$ now has the parameter $\nu=\frac{\epsilon}{1-\epsilon}$, where ϵ is the average BER on the S-R link. This choice introduces suboptimality as the average BER is used instead of the instantaneous BER. Assuming equal probability of the transmitted symbol $x_s=\{1,-1\}$, average probability of error for the ML-DF cooperative diversity system can be expressed as

$$P_e=\sum_{x_r\in\{1,-1\}}\epsilon^{\frac{1-x_r}{2}}(1-\epsilon)^{\frac{1+x_r}{2}}\times P\left(X+f(Y)<0|x_s=1,x_r\right).\quad(9)$$

IV. BER ANALYSIS

The conditional probability in (9) can be expressed as

$$P\left(X+f(Y)<0|x_s=1,x_r\right)=\int_{-\infty}^{\infty}F_{f(Y)}(-x)p_X(x)dx.\quad(10)$$

Substituting for $p_X(x)$ from Theorem 2.1 and $F_{f(Y)}(-x)$ from Corollary 2.1.1, in the above,

$$P\left(X+f(Y)<0|x_s=1,x_r\right)=\mathcal{I}_{m_s}\left(\frac{a_s}{\sqrt{b_s}},0,c_s\right)-\frac{1}{\sqrt{b_s}}\sum_{\upsilon\in\{1,-1\}}\upsilon\times\int_0^{\ln\frac{1}{\nu}}\mathcal{I}_{m_r}\left(\frac{\upsilon a_r}{\sqrt{b_r}},-\frac{\ln g(x)}{\sqrt{b_r}},c_r\right)\times\mathcal{J}_{m_s}\left(-\frac{\upsilon a_s}{\sqrt{b_s}},\frac{x}{\sqrt{b_s}},c_s\right)dx\quad(11)$$

after a change of variables. Since $\nu \ll 1$, from Proposition 2.1 and (11), we obtain

$$
\begin{aligned}
&P\left(X+f(Y)<0|x_s=1,x_r\right) \approx \\
&\qquad \mathcal{I}_{m_s}\left(\frac{a_s}{\sqrt{b_s}},0,c_s\right)-\frac{1}{\sqrt{b_s}}\sum_{\upsilon\in\{1,-1\}}\upsilon \\
&\qquad \times\int_0^{\ln\frac{1}{\nu}}\mathcal{I}_{m_r}\left(\frac{\upsilon a_r}{\sqrt{b_r}},\frac{x}{\sqrt{b_r}},c_r\right) \\
&\qquad \times\mathcal{J}_{m_s}\left(-\frac{\upsilon a_s}{\sqrt{b_s}},\frac{x}{\sqrt{b_s}},c_s\right)dx, \qquad (12)
\end{aligned}
$$

which can be expressed in closed form as (13) (see Appendix C) where $K=\sum_{k=1}^{m_r-1}l_k$, $N=\sum_{n=1}^{m_s-1}L_n$ and $\kappa_i=\frac{\sqrt{a_i^2+2b_ic_i}+\upsilon a_i}{b_i}$, $\mu_i=\frac{\sqrt{a_i^2+2b_ic_i}}{b_i}$ for $i\in\{r,s\}$. Substituting (13) in (9) results in a closed form expression for the average BER. Note that (11) yields an exact expression for the average BER expressed as an integral. The skewness between (11) and (13) is dependent on the BER for the S-R link, as evident from (2).

Theorem 4.1: The PL [4] and *exponential* approximations are equivalent

Proof: See Appendix D.

Thus, (13) is a closed form expression for the PL combiner.

Theorem 4.2: The diversity order [8] for ML-DF cooperation in Nakagami-m fading is

$$d=m_s+\min\left(m,m_r\right), \qquad (14)$$

where m,m_s,m_r are the fading figures on the S-R, S-D and R-D links.

Proof: See Appendix E.

V. Results and Discussion

Let $\bar{\gamma}=\frac{\Omega E_s}{N_0},\bar{\gamma}_s=\frac{\Omega_s E_s}{N_0},\bar{\gamma}_r=\frac{\Omega_r E_r}{N_0}$ and (m,m_s,m_r) be the respective fading parameters on the S-R, S-D and R-D links. For convenience, we assume $\bar{\gamma}=\bar{\gamma}_s$ and define $\frac{\bar{\gamma}_r}{\bar{\gamma}_s}=\xi$.

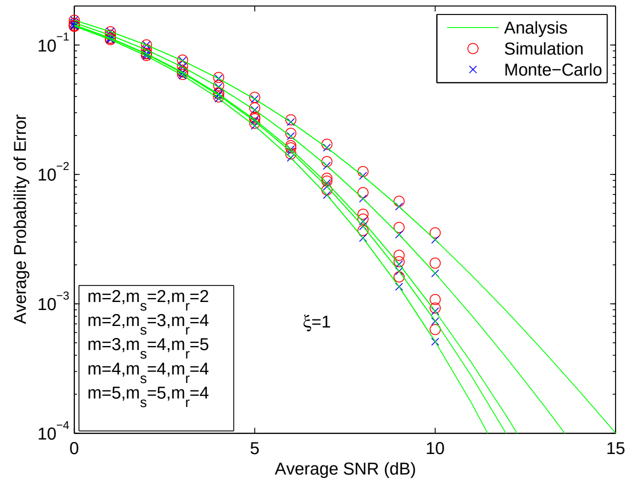

Fig. 2. BER simulation and analytical results. Sequence of plots in the same order as the parameters listed in the box.

For the system in Figure 1, actual BER simulations, monte-carlo simulations for (11), and the closed form approximation in (13) are compared in Figure 2, for different combinations of (m,m_s,m_r) and $\xi=1$. Simulation results closely follow the analysis curves, validating the BER expressions obtained in the paper. The BER keeps improving with higher Nakagami severities, as expected.

A comparison of the BER for ML-DF, simple adaptive DF [3] and the traditional two antenna system is presented in Figure 3 for (m,m_s,m_r) = (2,3,2) and (3,5,4), $\xi=1$. In both cases, the BER curves for ML-DF and simple adaptive DF are found to be parallel in the high SNR region, indicating a similar diversity order. However, ML-DF outperforms simple adaptive DF. The traditional two antenna system performs better than both cooperative schemes, as expected. For (2,3,2),

$$
\begin{aligned}
&P\left(X+f(Y)<0|x_s=1,x_r\right)=\mathcal{I}_{m_s}\left(\frac{a_s}{\sqrt{b_s}},0,c_s\right)+\frac{c_r^{m_r}c_s^{m_s}}{b_r^{m_r}b_s^{m_s}\mu_r^{2m_r-1}\mu_s^{2m_s-1}} \\
&\quad\times\sum_{\upsilon\in\{1,-1\}}\frac{\upsilon}{\kappa_r}\sum_{\mathbf{l}.\mathbf{v}_{m_r-1}=m_r-1}\prod_{k=1}^{m_r-1}\left[\frac{(2k-3)!!}{k!}\right]^{l_k}\frac{1}{l_k!}\sum_{\mathbf{L}.\mathbf{v}_{m_s-1}=m_s-1}\prod_{n=1}^{m_s-1}\left[\frac{(2k-3)!!}{n!}\right]^{L_n}\frac{1}{L_n!} \\
&\quad\times\sum_{\mathbf{q}.\mathbf{v}_K=K}\frac{\mu_r^K K!}{q_1!}\prod_{j=2}^{K}\left[\frac{1}{\kappa_r^j}+\frac{1}{\mu_r^j}\right]^{q_j}\frac{1}{j^{q_j}q_j!}\sum_{\mathbf{Q}.\mathbf{v}_N=N}\frac{N!\mu_s^N}{Q_1!}\prod_{i=2}^{N}\left[\frac{1}{\kappa_s^i}+\frac{1}{\mu_s^i}\right]^{Q_i}\frac{1}{i^{Q_i}Q_i!} \\
&\quad\times\Bigg[\delta_{Q_1}\sum_{\rho=0}^{q_1}\binom{q_1}{\rho}(\kappa_r+\kappa_s)^{-\rho-1}\left(\frac{1}{\kappa_r}+\frac{1}{\mu_r}\right)^{q_1-\rho}\Gamma\left(\rho+1,(\kappa_r+\kappa_s)\ln\frac{1}{\nu}\right) \\
&\quad+(1-\delta_{Q_1})\sum_{\rho=0}^{q_1}\sum_{\sigma=0}^{Q_1-1}\binom{q_1}{\rho}\binom{Q_1-1}{\sigma}(\kappa_r+\kappa_s)^{-\rho-\sigma-2}\left(\frac{1}{\kappa_s}+\frac{1}{\mu_s}\right)^{Q_1-1-\sigma} \\
&\quad\times\left(\frac{1}{\kappa_r}+\frac{1}{\mu_r}\right)^{q_1-\rho}\Bigg\{(\kappa_r+\kappa_s)\left(\frac{1}{\mu_s}-\frac{Q_1-1}{\kappa_s}\right) \\
&\quad\times\Gamma\left(\rho+\sigma+1,(\kappa_r+\kappa_s)\ln\frac{1}{\nu}\right)+\Gamma\left(\rho+\sigma+2,(\kappa_r+\kappa_s)\ln\frac{1}{\nu}\right)\Bigg\}\Bigg]. \qquad (13)
\end{aligned}
$$

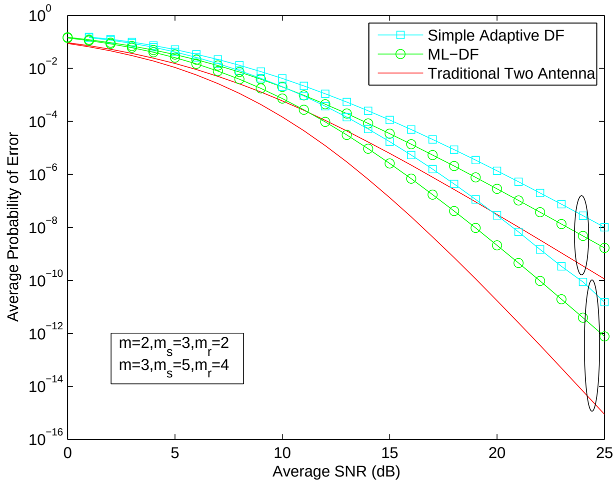

Fig. 3. ML-DF performs better than simple adaptive DF. Sequence of plots in same order as parameters listed in the box.

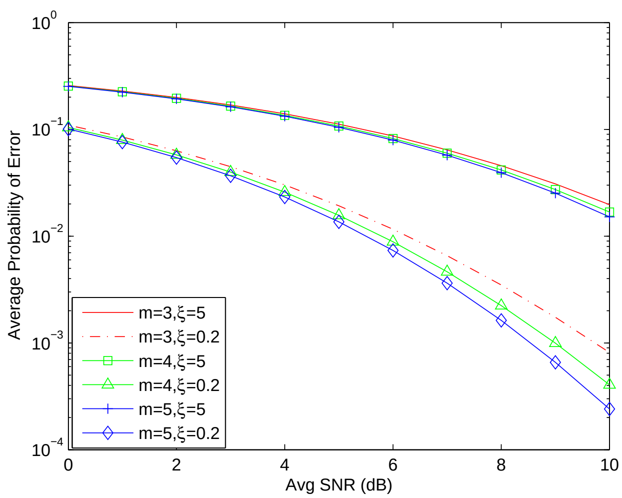

Fig. 4. Variation of the BER with m and ξ. System performance improves when $\xi < 1$.

the two antenna system is seen to have a similar diversity order as the two cooperative schemes. In contrast, for (3,5,4), in the high SNR region, the slope of the BER curve for the two antenna system falls faster when compared to the cooperative systems, indicating a higher diversity order, validating (14).

In Figure 4, we investigate the consequences of the link SNR imbalance for similar Nakagami severities, i.e. $m = m_s = m_r$. ξ has been chosen to be 0.2 or 5. For $\xi < 1$, we find that the BER improves significantly. This is because of improved performance on the S-R as well as the S-D link as a consequence of increased source power. For $\xi > 1$, system performance is relatively poor due to the weak S-R link which results in higher decode errors at the relay. These errors are then propagated to the destination because of a strong R-D link resulting from an increase in relay power.

VI. Conclusions

In this paper, we have obtained closed form expressions for the statistics of gamma CG random variables. These results were then applied to BER analysis for an ML-DF cooperative diversity system employing a single relay, for the Nakagami-m fading channel. The exact BER for ML-DF was expressed in terms of a single integral, while a closed form expression for the BER was obtained through the exponential approximation and shown to be the BER for the PL combiner. A measure of the diversity order for ML-DF cooperation was also obtained. Finally, all analytical results were validated through simulations. Numerical results indicate that ML-DF is superior to simple adaptive DF though both have a similar diversity order.

Appendix A

From [5], [9], we obtain

$$\int_0^\infty x^{m-1} Q\left(\frac{ax+b}{\sqrt{x}}\right) e^{-cx} dx = (-1)^{m-1} \frac{d^{m-1}}{dc^{m-1}} \left[\frac{\exp\left(-b\left(a+\sqrt{a^2+2c}\right)\right)}{\sqrt{a^2+2c}\left(a+\sqrt{a^2+2c}\right)} \right]. \quad (15)$$

Let $G(x) = \frac{e^{-bx}}{x(x-a)}$, $\quad H(x) = a + \sqrt{a^2+2x}$, so that

$$G(H(c)) = \frac{\exp\left(-b\left(a+\sqrt{a^2+2c}\right)\right)}{\sqrt{a^2+2c}\left(a+\sqrt{a^2+2c}\right)}. \quad (16)$$

Using the Fàa Di Bruno formula [10], the nth derivative of $G(x)$ can be expressed as

$$G^{(n)}(x) = (-1)^n n! G(x) \times \sum_{\mathbf{q}.\mathbf{v}_n = n} \prod_{j=1}^{n} \left[b\delta_{j-1} + \frac{1}{x^j} + \frac{1}{(x-a)^j} \right]^{q_j} \frac{1}{j^{q_j} q_j!}. \quad (17)$$

The kth derivative of $H(x)$ is

$$H^{(k)}(x) = (-1)^{k-1}(2k-3)!!(a^2+2x)^{\frac{1}{2}-k}. \quad (18)$$

Using the Fàa Di Bruno formula again to obtain the $m-1$th derivative of $G(H(c))$ from (17) and (18), after some algebra, (4) can be obtained from (15) and (16).

Appendix B

The CDF of Z can be expressed as [5]

$$F_Z(z) = \begin{cases} 1 - \int_0^\infty \frac{c^m x^{m-1}}{\Gamma(m)} Q\left(\frac{-ax+z}{\sqrt{bx}}\right) e^{-cx} dx & z \geq 0 \\ \int_0^\infty \frac{c^m x^{m-1}}{\Gamma(m)} Q\left(\frac{ax-z}{\sqrt{bx}}\right) e^{-cx} & z < 0 \end{cases}, \quad (19)$$

where $\Gamma(\cdot)$ denotes the gamma function [9]. Now, applying Lemma 2.1, we obtain the CDF in (6). Differentiating the CDF results in the PDF. (7) is trivially obtained from (6) using the approach in [5]. The Nth moment of Z can be expressed as

$$E[Z^N] = \int_0^\infty x^N dF_Z(x) dx + \int_{-\infty}^0 x^N dF_Z(x) dx. \quad (20)$$

The primitive of the first integrand can be expressed as

$$\int x^N dF_Z(x) = x^N F_Z(x) - N \int \left(1 - I_m\left(-\frac{a}{\sqrt{b}}, \frac{x}{\sqrt{b}}, c\right)\right) x^{N-1} dx, \quad (21)$$

which, on simplification and applying limits, yields

$$\int_0^\infty x^N dF_Z(x) = N \int_0^\infty x^{N-1} I_m\left(-\frac{a}{\sqrt{b}}, \frac{x}{\sqrt{b}}, c\right) dx. \quad (22)$$

Similarly, after appropriate change of variables,

$$\int_{-\infty}^0 x^N dF_Z(x) = (-1)^N N \int_0^\infty x^{N-1} I_m\left(\frac{a}{\sqrt{b}}, \frac{x}{\sqrt{b}}, c\right) dx. \quad (23)$$

From (22) and (23), (20) can be expressed as

$$E[Z^N] = N \sum_{\upsilon\in\{1,-1\}} \upsilon^N \int_0^\infty x^{N-1} I_m\left(-\frac{\upsilon a}{\sqrt{b}}, \frac{x}{\sqrt{b}}, c\right) dx. \quad (24)$$

From (4), it is obvious that to evaluate the above integral, we need to compute the integral

$$\Psi = \int_0^\infty x^{N-1}\left(x + \frac{1}{\kappa_Z} + \frac{1}{\mu_Z}\right)^{q_1} e^{-\kappa_Z x} dx, \quad (25)$$

which, following the binomial expansion and integrating, can be expressed in the closed form

$$\Psi = \sum_{\rho=0}^{q_1} \binom{q_1}{\rho} {\kappa_Z}^{-N-\rho} \left(\frac{1}{\kappa_Z} + \frac{1}{\mu_Z}\right)^{q_1-\rho} \Gamma\left(N+\rho\right). \quad (26)$$

From (4), (24) and (26), we obtain (8).

Appendix C

From (4) and (5), it is obvious that to evaluate (12), the integrals

$$\Upsilon = \int_0^{\ln\frac{1}{\nu}} e^{-(\kappa_r+\kappa_s)x}\left[x + \frac{1}{\kappa_r} + \frac{1}{\mu_r}\right]^{q_1} \times \left[\frac{Q_1-1}{\kappa_s} - \left(x + \frac{1}{\mu_s}\right)\right] \times \left[x + \frac{1}{\kappa_s} + \frac{1}{\mu_s}\right]^{Q_1-1} dx, \quad (27)$$

$$\Delta = \int_0^{\ln\frac{1}{\nu}} e^{-(\kappa_r+\kappa_s)x}\left[x + \frac{1}{\kappa_r} + \frac{1}{\mu_r}\right]^{q_1} dx \quad (28)$$

need to be evaluated. The indices q_1 and Q_1 result from the expressions for $\mathcal{I}_{m_r}$ and $\mathcal{J}_{m_s}$ respectively. Using the binomial expansion and rearranging (27), we obtain

$$\Upsilon = \sum_{\rho=0}^{q_1}\sum_{\sigma=0}^{Q_1-1}\binom{q_1}{\rho}\binom{Q_1-1}{\sigma}\left(\frac{1}{\kappa_r}+\frac{1}{\mu_r}\right)^{q_1-\rho} \times \left(\frac{1}{\kappa_s}+\frac{1}{\mu_s}\right)^{Q_1-1-\sigma}\left\{\left(\frac{Q_1-1}{\kappa_s}-\frac{1}{\mu_s}\right) \times \int_0^{\ln\frac{1}{\nu}} x^{\rho+\sigma}e^{-(\kappa_r+\kappa_s)x}dx - \int_0^{\ln\frac{1}{\nu}} x^{\rho+\sigma+1}e^{-(\kappa_r+\kappa_s)x}dx\right\}, \quad (29)$$

which can be expressed in closed form as

$$\Upsilon = \sum_{\rho=0}^{q_1}\sum_{\sigma=0}^{Q_1-1}\binom{q_1}{\rho}\binom{Q_1-1}{\sigma}(\kappa_r+\kappa_s)^{-\rho-\sigma-2} \times \left(\frac{1}{\kappa_r}+\frac{1}{\mu_r}\right)^{q_1-\rho}\left(\frac{1}{\kappa_s}+\frac{1}{\mu_s}\right)^{Q_1-1-\sigma} \times\left\{(\kappa_r+\kappa_s)\left(\frac{Q_1-1}{\kappa_s}-\frac{1}{\mu_s}\right) \times\Gamma\left(\rho+\sigma+1,(\kappa_r+\kappa_s)\ln\frac{1}{\nu}\right) -\Gamma\left(\rho+\sigma+2,(\kappa_r+\kappa_s)\ln\frac{1}{\nu}\right)\right\}, \quad (30)$$

where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function [9, (8.350.1)]. Following a similar approach, (28) can be can be obtained in closed form as

$$\Delta = \sum_{\rho=0}^{q_1}\binom{q_1}{\rho}(\kappa_r+\kappa_s)^{-\rho-1}\left(\frac{1}{\kappa_r}+\frac{1}{\mu_r}\right)^{q_1-\rho} \times \Gamma\left(\rho+1,(\kappa_r+\kappa_s)\ln\frac{1}{\nu}\right). \quad (31)$$

From (30) and (31), (12) can be expressed as (13).

Appendix D

The conditional probability in (9) can also be expressed as

$$P\left(X+f(Y)<0|x_s=1,x_r\right) = \int_{-\infty}^{\infty} F_X\left(-f(x)\right)p_Y(x)dx, \quad (32)$$

which, from the PL approximation [4] results in

$$P\left(X+f(Y)<0|x_s=1,x_r\right) \approx F_X\left(\ln\nu\right)\left[1-F_Y\left(\ln\frac{1}{\nu}\right)\right] + F_X\left(\ln\frac{1}{\nu}\right)F_Y\left(\ln\nu\right) + \int_{\ln\nu}^{\ln\frac{1}{\nu}} F_X\left(-x\right)p_Y(x)dx. \quad (33)$$

In the above, the integral can be expressed as

$$\int_{\ln\nu}^{\ln\frac{1}{\nu}} F_X\left(-x\right)p_Y(x)dx = \int_{\ln\nu}^{\ln\frac{1}{\nu}} F_X\left(-x\right)dF_Y(x), \quad (34)$$

which, upon integration by parts, yields

$$\int_{\ln\nu}^{\ln\frac{1}{\nu}} F_X\left(-x\right)p_Y(x)dx = \left[F_X\left(-x\right)F_Y(x)\right]_{\ln\nu}^{\ln\frac{1}{\nu}} + \int_{\ln\nu}^{\ln\frac{1}{\nu}} F_Y(-x)p_X\left(x\right)dx \quad (35)$$

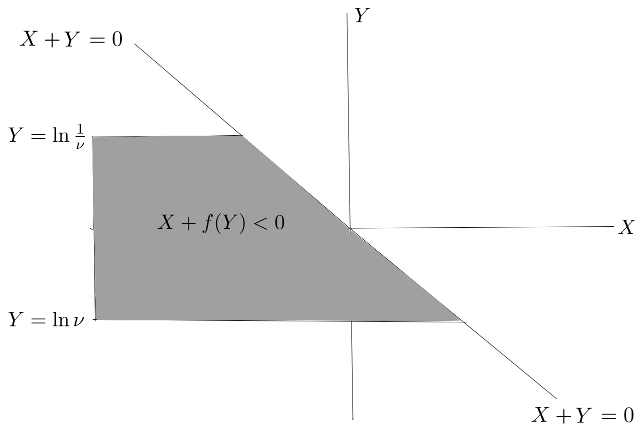

Fig. 5. From the PL approximation, $\{X+f(Y)<0\} \subset \{X+Y<0\}$.

after a change of variables. Substituting from (6) in the above,

$$\begin{aligned}\int_{\ln\nu}^{\ln\frac{1}{\nu}} F_Y(-x)p_X(x)\,dx &= F_X(0) - F_X(\ln\nu)\\ &+ \frac{1}{\sqrt{b_s}}\left[\int_{\ln\nu}^{0} \mathcal{I}_{m_r}\left(-\frac{a_r}{\sqrt{b_r}}, -\frac{x}{\sqrt{b_r}}, c_r\right)\right.\\ &\times\mathcal{J}_{m_s}\left(\frac{a_s}{\sqrt{b_s}}, -\frac{z}{\sqrt{b_s}}, c_s\right)dx\\ &- \int_{0}^{\ln\frac{1}{\nu}} \mathcal{I}_{m_r}\left(\frac{a_r}{\sqrt{b_r}}, \frac{x}{\sqrt{b_r}}, c_r\right)\\ &\left.\times\mathcal{J}_{m_s}\left(-\frac{a_s}{\sqrt{b_s}}, \frac{z}{\sqrt{b_s}}, c_s\right)dx\right]. \quad (36)\end{aligned}$$

From (33), (35) and (36), we obtain (12), and the proof is complete.

APPENDIX E

Since $\epsilon \approx \binom{2m-1}{m}\left(\frac{1}{4\bar{\gamma}}\right)^m \ll 1$ for $\bar{\gamma} \gg 1$ [8, 14-4-18], from (9),

$$\begin{aligned}P_e \leq\, &\epsilon P\left(X+f(Y)<0|x_s=1, x_r=-1\right)\\ &+P\left(X+f(Y)<0|x_s=1, x_r=1\right),\end{aligned} \quad (37)$$

where m and $\bar{\gamma}$ are the fading parameters on the S-R link. From Figure 5, where the PL approximation [4] is invoked for $f(\cdot)$ and [11, (10)], defining $\bar{\gamma}_i = \frac{\Omega_i E_i}{N_0}, i \in \{r, s\}$,

$$P\left(X+f(Y)<0|x_s=1, x_r=1\right) \leq \frac{1}{2}\left(\frac{m_s}{\bar{\gamma}_s}\right)^{m_s}\left(\frac{m_r}{\bar{\gamma}_r}\right)^{m_r}. \quad (38)$$

For $x_r=-1$, from (33) and (35),

$$\begin{aligned}&P(X+f(Y)<0|x_s=1, x_r=-1)\\ &\qquad= F_X\left(\ln\nu\right) + \int_{\ln\nu}^{\ln\frac{1}{\nu}} F_Y(-x)p_X(x)\,dx\\ &\qquad\leq F_X\left(\ln\frac{1}{\nu}\right)\end{aligned} \quad (39)$$

$\because F_Y(\cdot) \leq 1$. Since $X \sim \mathcal{N}\left(4\gamma_s, 8\gamma_s\right), \gamma_s = \frac{E_s}{N_0}h_s^2$ and $\ln\frac{1}{\nu} = \ln\left(\frac{1}{\epsilon}-1\right) \approx \ln\frac{1}{\epsilon}$,

$$\begin{aligned}F_X\left(\ln\frac{1}{\nu}\right) &= Q\left(\frac{4\gamma_s - \ln\frac{1}{\nu}}{2\sqrt{2\gamma_s}}\right)\\ &\approx Q\left(\sqrt{2\gamma_s} + \frac{\ln\epsilon}{2\sqrt{2\gamma_s}}\right) \quad (40)\\ &\approx Q\left(\sqrt{2\gamma_s}\right)\\ &\approx \binom{2m_s-1}{m_s}\left(\frac{1}{4\bar{\gamma}_s}\right)^{m_s} \quad (41)\end{aligned}$$

using the high SNR approximation in (40). Thus, from (37), (38), (39) and (41), we obtain (14).

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