Performance Analysis of Maximum Likelihood Decode and Forward Cooperative Systems in Rayleigh Fading

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***Abstract*—The bit error rate (BER) performance analysis of maximum-likelihood (ML) based decode and forward (DF) cooperative diversity systems has been a subject of considerable interest. Exact analysis of ML-DF transmission has been considered a challenging problem due to the nonlinear characteristic of the ML detector. In this paper, we provide exact expressions for the BER of ML-DF cooperative systems employing a single relay. We extend these results to the case of multiple relays for the piecewise linear (PL) combiner, that is known to be a close approximation of the ML detector. This is done by using a novel theory of conditionally Gaussian random variables. By expressing the ML decision variable in terms of functions of conditionally Gaussian variables, exact expressions for the BER of the ML-DF system are obtained. Through simulation results, we verify the validity of the derived analytical expressions.**

***Index Terms*—Cooperative diversity, conditionally Gaussian, ML, PL.**

I. Introduction

The ML decision rule for DF cooperative systems [1]–[8] was first presented in [3] followed by a detailed derivation in [4] for both coherent and noncoherent demodulation. Further, the PL combiner was suggested as a useful practical alternative for the ML detector in [4]. For a Rayleigh faded channel, BER analysis for noncoherent demodulation for a single relay DF system using the PL combiner was presented in [5]. The equivalent problem for coherent detection was addressed in [6] for the Gaussian fading channel, again for the PL combiner. While all the above attempts at obtaining expressions for the BER for coherent and noncoherent demodulation are remarkable, the approach used in both methods is direct and results in extremely complicated expressions for the BER.

The key contributions of this paper are i) exact expressions for the BER for ML detector and PL combiner for a single relay and ii) tight bound on the BER for PL combiner with multiple relays, for a coherent DF cooperative system in Rayleigh fading. This is done by using a novel concept of conditionally Gaussian random variables, which appear in the ML decision rule for coherent demodulation for DF cooperative diversity systems. Though these variables appear in the decision rules for Rayleigh fading channels [9], to the best of our knowledge, they have never been discussed in the available literature. The only approximation that we make at the system level is based on [5] and [6], where the average BER for the source-relay link is used in the ML decision rule instead of the instantaneous BER.

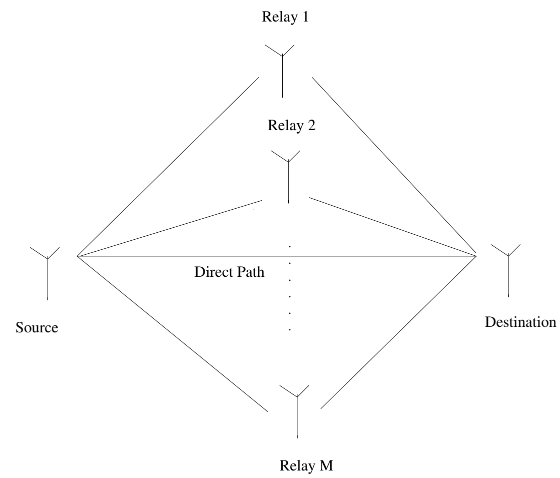

Fig. 1. Cooperative diversity system with M relays.

II. System Model

We consider the model in [5] with multiple relays (R) between the source (S) and destination (D) as shown in Fig. 1. The DF method is employed at the relay followed by ML decision at the destination. Assuming that the modulation is binary phase shift keying (BPSK) and all transmissions are on orthogonal channels, let $x_s \in \{1, -1\}$ and $x_r \in \{1, -1\}$ be the bits transmitted at the source and relay with powers E_s and E_r respectively. The received symbols on the S-D, S-R and R-D links are respectively

$$\begin{aligned} y_{d,s} &= \sqrt{E_s} h_{d,s} x_s + z_{d,s}, \\ y_{r,s} &= \sqrt{E_s} h_{r,s} x_s + z_{r,s}, \\ y_{d,r} &= \sqrt{E_r} h_{d,r} x_r + z_{d,r}, \end{aligned} \tag{1}$$

for $r = 1, 2, \ldots, M$. Also, $z_{d,s}, z_{r,s}, z_{d,r} \sim \mathcal{CN}(0, N_0)$ represent the additive white Gaussian noise (AWGN) at the respective relays and the destination. The channel experiences Rayleigh fading, hence the fading coefficients $h_{d,s} \sim \mathcal{CN}(0, \Omega_{d,s}), h_{r,s} \sim \mathcal{CN}(0, \Omega_{r,s})$ and $h_{d,r} \sim \mathcal{CN}(0, \Omega_{d,r})$ are zero mean complex circularly Gaussian.

A. ML Decision

The ML decision criterion at the destination for BPSK modulation may be obtained from [4], [5] as

$$X + \sum_{r=1}^{M} f(Y_r) \underset{-1}{\overset{1}{\gtrless}} 0, \tag{2}$$

where

$$X = \frac{4\sqrt{E_s}\text{Re}\{h_{d,s}^* y_{d,s}\}}{N_0}, Y_r = \frac{4\sqrt{E_s}\text{Re}\{h_{d,r}^* y_{d,r}\}}{N_0}, \tag{3}$$

$$f(t) = \ln \frac{\delta + e^t}{1 + \delta e^t}, \quad 0 < \delta < 1 \tag{4}$$

and $\{^*\}$ denotes the complex conjugate operation. We would like to mention that $f(Y_r)$ now has the parameter $\delta_r = \frac{\epsilon_r}{1-\epsilon_r}$ for the rth relay, where ϵ_r is the BER for the S-R link. To simplify the analysis, we use the suboptimal ML scheme proposed in [6] which results in $\epsilon_r = \frac{1}{2}\left[1 - \left(1 + \frac{\Omega_{r,s}E_s}{N_0}\right)^{-\frac{1}{2}}\right]$.

B. PL combiner

Lemma 2.1: The function $f(t)$ in (4) has the following piecewise linear approximation [4], [5]

$$f(t) \approx \begin{cases} \ln \frac{1}{\delta} & t \geq \ln \frac{1}{\delta} \\ t & \ln \delta < t < \ln \frac{1}{\delta} \\ \ln \delta & t < \ln \delta \end{cases}. \tag{5}$$

Using the above approximation in (2), we obtain the PL combiner for ML-DF cooperative systems.

C. Problem Definition

Assuming equal probability of the symbols $x_s = \{1, -1\}$, the average probability of error for the ML-DF cooperative diversity system can be expressed as

$$\begin{aligned} P_e &= \sum_{\mathbf{x}} \prod_{r=1}^{M} \epsilon_r^{\frac{1-x_r}{2}} (1-\epsilon_r)^{\frac{1+x_r}{2}} \\ &\times P\left(X + \sum_{i=1}^{M} f(Y_i) < 0 | x_s = 1, \mathbf{x}\right), \end{aligned} \tag{6}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_M)$ is the set of all possible $M-$tuples formed by the symbols transmitted by the relays. We wish to find out a closed form expression for P_e in (6).

D. Solution Strategy

From (6), it is clear that the knowledge of the statistics of X and $f(Y_r)$ defined in (3) and (4) would greatly simplify the analysis. The variable X can be expressed using (1) as

$$X = \frac{4E_s}{N_0} x_s |h_{d,s}|^2 + \frac{4\sqrt{E_s}}{N_0} Re(h_{d,s}^* z_{d,s}) \tag{7}$$

The conditional mean and variance of X can be expressed as

$$\begin{aligned} E[X|h_{d,s}] &= \frac{4E_s}{N_0} x_s |h_{d,s}|^2 \\ var(X|h_{d,s}) &= \frac{8E_s}{N_0} |h_{d,s}|^2 \end{aligned} \tag{8}$$

From (7) and (8), we find that the variable $X \sim \mathcal{N}\left(\frac{4}{N_0}\sqrt{E_s} x_s |h_{d,s}|^2, \frac{8}{N_0}|h_{d,s}|^2\right)$ where $|h_{d,s}|^2$ is exponential with parameter $\frac{1}{\Omega_{d,s}}$. It is easy to show that even Y_r has a similar distribution. We call such distributions as conditionally Gaussian. In the following section, we find the cumulative distribution function (CDF), probability density function (PDF) and characteristic function (CF) of conditionally Gaussian random variables like X, Y_r and their functions $f(Y_r)$. Using these, we find a closed form expression for P_e in (6) for both the ML detector as well as the PL combiner in the subsequent sections.

III. Conditionally Gaussian Distributions

A. Preliminaries

Definition 3.1: A conditionally Gaussian random variable with parameters $a, b > 0$ is defined as $Z \sim \mathcal{N}(aA, bA)$ where A is a random variable. In this paper, we assume that A is exponentially distributed with parameter c. We refer to such variables as *exponential conditionally Gaussian random variables*.

Definition 3.2: For $0 < \delta < 1$, we define the function

$$s(t) = \frac{e^{-t} - \delta}{1 - \delta e^{-t}}. \tag{9}$$

It directly follows from the above that $s(t) \leq e^{-t}, t > 0$. In the rest of the paper, we use the symbols f and s to represent the functions defined in (4) and (9) respectively.

Lemma 3.1: For any constants $a, b > 0$ and $c > 0$,

$$\int_0^\infty Q\left(\frac{ax + b}{\sqrt{x}}\right) e^{-cx} dx = \frac{\exp\left(-b\left(a + \sqrt{a^2 + 2c}\right)\right)}{\sqrt{a^2 + 2c}\left(a + \sqrt{a^2 + 2c}\right)}, \tag{10}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$.

Proof: Defining the error function $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and noting that $Q(x) = \frac{1}{2}\left[1 - \Phi\left(\frac{x}{\sqrt{2}}\right)\right]$, (10) is obtained using the result in [10, 6.297.1].

Lemma 3.2: For any constants $a > 0, b$ and $c > 0$,

$$\frac{1}{\sqrt{2\pi}c} \int_0^\infty e^{-ax} e^{-\frac{(x-b)^2}{2c^2}} dx = e^{-ab + \frac{b^2 c^2}{2}} Q\left(ac - \frac{b}{c}\right) \tag{11}$$

Proof: Completing squares and using [10, 3.322.2], we obtain (11).

Definition 3.3: We define the integral

$$\Lambda(p, q) = \frac{1}{p} \int_0^{\ln \frac{1}{\delta}} (s(x))^p e^{-qx} dx. \tag{12}$$

Since $s(t) \leq e^{-t}, t > 0$, we have

Lemma 3.3:

$$\Lambda(p, q) \approx \frac{1 - \delta^{p+q}}{p(p+q)}, \quad \delta << 1. \tag{13}$$

B. Closed form expressions for the statistical functions of conditionally Gaussian and related distributions

Theorem 3.1: For any constants $a, b > 0$ and $c > 0$, if $Z \sim \mathcal{N}(aA, bA)$ be conditionally Gaussian and A be exponentially distributed with parameter c, we have the following expressions for the CDF, PDF and the CF of Z.

$CDF:$

$$F_Z(z) = \begin{cases} 1 - \frac{g e^{-zh(-1)}}{h(-1)} & z \geq 0 \\ \frac{g e^{zh(1)}}{h(1)} & z < 0 \end{cases} \tag{14}$$

$PDF:$

$$p_Z(z) = \begin{cases} g e^{-zh(-1)} & z \geq 0 \\ g e^{zh(1)} & z < 0 \end{cases} \tag{15}$$

$CF:$

$$\phi_Z(t) = \frac{g\left(h(1) + h(-1)\right)}{(h(-1) - jt)(h(1) + jt)}, \tag{16}$$

where

$$h(m) = \frac{\sqrt{a^2 + 2bc} + ma}{b}, \quad m \in \{1, -1\}$$

$$g = \frac{c}{\sqrt{a^2 + 2bc}} \quad \text{and} \quad \frac{1}{g} = \frac{1}{h(1)} + \frac{1}{h(-1)} \tag{17}$$

Proof: See Appendix.

Theorem 3.2: The CDF of $V = f(Z)$, for Z defined in Theorem 3.1 is given by

$$F_V(v) = \begin{cases} 1 & v \geq \ln \frac{1}{\delta} \\ 1 - \frac{g}{h(-1)} s(v)^{h(-1)} & 0 \leq v < \ln \frac{1}{\delta} \\ \frac{g}{h(1)} s(v)^{-h(1)} & \ln \delta \leq v < 0 \\ 0 & v < \ln \delta \end{cases} \tag{18}$$

Defining $\phi(x, t) = \frac{t(1-\delta^{x+t})}{x(x+t)}$ the CF of V for the PL approximation is given by

$$\phi_V(t) = \left[1 - g\phi(h(1), jt) - g\phi(h(-1), -jt)\right]. \tag{19}$$

Proof: See Appendix.

C. Conditionally Gaussian distributions in ML-DF cooperative systems

From the discussion in section II.D and Definition 3.1, we have the following lemma.

Lemma 3.4: Let X and $\{Y_r\}_{r=1}^M$ be as defined in (3). Then X and Y_r are conditionally Gaussian with parameters $a = \frac{4E_s}{N_0} x_s, b = \frac{8E_s}{N_0}, c = \frac{1}{\Omega_{d,s}}$ and $a_r = \frac{4E_r}{N_0} x_r, b_r = \frac{8E_r}{N_0}, c_r = \frac{1}{\Omega_{d,r}}$ respectively.

IV. BER FOR ML-DF COOPERATIVE SYSTEMS

Using the Gil-Pelaez theorem [11], the conditional probability in (6) can be expressed as

$$P\left(X + \sum_{r=1}^{M} f(Y_r) < 0 | x_s = 1, \mathbf{x}\right) = \frac{1}{2} - \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\phi_X(t)}{t} \prod_{r=1}^{M} \phi_{f(Y_r)}(t) dt, \tag{20}$$

assuming that X and Y_r are mutually independent and denoting the CFs of X and Y_r as $\phi_X(t)$ and $\phi_{f(Y_r)}(t)$ respectively. While $\phi_X(t)$ is known, using Lemma 3.4 and Theorem 3.1, a closed form expression for $\phi_{f(Y_r)}(t)$ is available only for the PL combiner from (19), Theorem 3.2 and we have to fall back on the CDF approach to obtain an exact expression for the BER of the ML detector.

A. Exact Analysis for single relay

Theorem 4.1: (ML Detector) For a single relay between the source and destination, the closed form expression for the conditional probability in (6) is given by

$$P(X + f(Y_1) < 0 | x_s = 1, x_1) = \frac{g}{h(1)} + g g_1 \times \left[\Lambda\left((h_1(1), h(-1)\right)\right] - \left[\Lambda\left((h_1(-1), h(1)\right)\right] \tag{21}$$

for Λ in Definition 3.3 and g, h and g_1, h_1 defined in (17) being the respective parameters for X and Y_1.

Corollary: The expression in (21) has the following approximation after dropping the subscript for δ

$$P(X + f(Y_1) < 0 | x_s = 1, x_1) = \frac{g}{h(1)} + g g_1 \times \sum_{m \in \{1, -1\}} \frac{(-1)^{\frac{1-m}{2}} \left(1 - \delta^{h_1(m) + h(-m)}\right)}{h_1(m)(h_1(m) + h(-m))}. \tag{22}$$

Proof: From (6), we have

$$P\left(X + f(Y_1) < 0 | x_s = 1, x_1\right) = F_{f(Y_1)}(-X). \tag{23}$$

From Theorem 3.2, we obtain

$$F_{f(Y_1)}(-X) = \begin{cases} 0 & X \geq \ln \frac{1}{\delta} \\ \frac{g_1}{h_1(1)} \left(s(X)\right)^{h_1(1)} & 0 \leq X < \ln \frac{1}{\delta} \\ 1 - \frac{g_1}{h_1(-1)} \left(s(X)\right)^{-h_1(-1)} & \ln \delta \leq X < 0 \\ 1 & X < \ln \delta \end{cases}.$$

The required probability may now be obtained by averaging the above over X. This results in $\int_{-\infty}^{\infty} F_{f(Y_1)}(-x) p_X(x) dx$, which upon substituting for $p_X(x)$ from Theorem 3.1, and using (12) gives (21). From Lemma 3.3, for $\delta << 1$, we obtain (22).

Theorem 4.2: (PL Combiner) The closed form expression for the conditional probability in (6) for the PL DF system is exactly the same as (22) in Corollary 4.1.

Proof: We have

$$P\left(X + f(Y_1) < 0 | x_s = 1, x_1\right) = F_X(-f(Y_1)). \tag{24}$$

From (24), using Theorem 3.1, we obtain

$$F_X(-f(Y_1)) = \begin{cases} 1 - \frac{g \exp[f(Y_1) h(-1)]}{h(-1)} & f(Y_1) < 0 \\ \frac{g \exp[-f(Y_1) h(1)]}{h(1)} & f(Y_1) \geq 0 \end{cases}. \tag{25}$$

Using the PL approximation from Lemma 2.1 in the above, we obtain

$$F_X(-f(Y_1)) = \begin{cases} 1 - \frac{g \delta^{h(-1)}}{h(-1)} & Y_1 < \ln \delta \\ 1 - \frac{g \exp[Y_1 h(-1)]}{h(-1)} & \ln \delta \leq Y_1 < 0 \\ \frac{g \exp[-Y_1 h(1)]}{h(1)} & 0 \leq Y_1 < \ln \frac{1}{\delta} \\ \frac{g \delta^{h(1)}}{h(1)} & Y_1 \geq \ln \frac{1}{\delta} \end{cases}. \tag{26}$$

where we have made use of the fact that $f(t) \lesseqqgtr 0 \Leftrightarrow t \lesseqqgtr 0$ for $0 < \delta < 1$. Averaging (26) over Y_1, the required probability may be expressed as $\int_{-\infty}^{\infty} F_X(-f(x)) p_{Y_1}(x) dx$ which, after substituting for $p_{Y_1}(x)$ from Theorem 3.1 and evaluating the integral and simplifying results in (22). Using Lemma 3.4 and substituting the expressions obtained from Theorems 4.1 and 4.2 in (6), a closed form expression for the BER for the DF cooperative diversity system is obtained for the ML detector and PL combiner respectively.

B. Central Limit Theorem approximation for multiple relays

A direct application of the complex Gil-Peleaz formula [12] has been the standard approach to finding expressions for the BER in wireless fading channels. However, the characteristic function $\phi_{f(Y_r)}(t)$ obtained in (19) is not well behaved for such an approach . Hence, we use the central limit theorem (CLT) [13] in (20) to find an expression for the BER when a large number of relays are used.

Lemma 4.1: The first and second moments of $f(Y_r)$ are

$$\mu_r = \alpha(-1) - \alpha(1),$$
$$E\left[(f(Y_r))^2\right] = \beta(-1) + \beta(1) \qquad (27)$$

where, for $m \in \{1, -1\}$

$$\alpha(m) = g_r \frac{1 - \delta^{h_r(m)}}{[h_r(m)]^2}$$
$$\beta(m) = g_r \frac{2\left(1 - \delta^{h_r(m)} + \delta^{h_r(m)} \ln \delta^{h_r(m)}\right)}{[h_r(m)]^3}. \qquad (28)$$

The variance of $f(Y_r)$ can be obtained as

$$\sigma_r^2 = E\left[(f(Y_r))^2\right] - \mu_r^2. \qquad (29)$$

Proof: Using the CF approach in [9, p.34], $E[(f(Y_r))^n] = (-j)^n \phi_{f(Y_r)}^{(n)}(0)$, where $\phi_{f(Y_r)}^{(n)}(t)$ is the nth derivative. Evaluating the derivative for $n = 1, 2$ gives (27).

Theorem 4.3: (CLT approximation) For large M and μ_r, σ_r defined in Lemma 4.1, the conditional probability defined in (6) can be expressed using the CLT as

$$P(X + Y < 0 | x_s = 1, \mathbf{x}) = Q\left(\frac{\mu}{\sigma}\right) + J(1) - J(-1) \qquad (30)$$

where $Y = \sum_{r=1}^{M} f(Y_r)$ and for $m \in \{1, -1\}$

$$J(m) = \frac{g}{h(m)} e^{-mh(m)\mu + \frac{[h(m)]^2\sigma^2}{2}} Q\left(\sigma h(m) - m\frac{\mu}{\sigma}\right)$$

for $\mu = \sum_{r=1}^{M} \mu_r, \sigma^2 = \sum_{r=1}^{M} \sigma_r^2$.

Proof: Using the central limit theorem, for large M, we have the distribution $Y \sim \mathcal{N}(\mu, \sigma^2)$. Since $P(X + Y < 0 | x_s = 1 | \mathbf{x}) = F_X(-Y)$, this probability may be expressed, using Theorem 3.1 as

$$F_X(-Y) = \begin{cases} 1 - \frac{g}{h(-1)} e^{h(-1)Y} & Y \le 0 \\ \frac{g}{h(1)} e^{-h(1)Y} & Y > 0 \end{cases} \qquad (31)$$

Unconditioning with respect to Y, the desired probability may be expressed as $\int_{-\infty}^{\infty} F_X(-y) p_Y(y) dy$, where $p_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(y-\mu)^2}{2\sigma^2}}, y \in (-\infty, \infty)$. Using Lemma 3.2 to evaluate this integral and simplifying, we obtain (30).

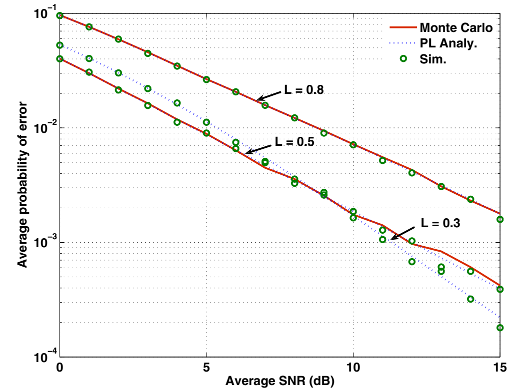

Fig. 2. Comparison of analytical and simulation results for $M = 1$.

V. Results and Discussion

The simulation setup is similar to the one used in [5]. The total power used by the system is assumed to be one unit, shared by the source and relays. This may be represented as $E_s + \sum_{r=1}^{M} E_r = 1$. Also, we define $L_r = \frac{\text{distance between source and } r\text{th relay}}{\text{distance between source and destination}}$ with the channel fading power between source to relay $\Omega_{r,s} \propto \frac{1}{L_r^4}$. A comparison of the simulation and analytical results for the case of a single relay ($M = 1$) is provided in Fig. 2 for $L = 0.3, 0.5$ and 0.8 and $E_s = E_r = 0.5$. For the analytical results, the closed form BER expression in (6) is used along with Theorem 4.1 and Theorem 4.2 for the ML detector and PL combiner respectively. Monte Carlo simulations have been used to evaluate the definite integral that appears in the conditional BER expression for the ML detector. As we can see from Fig. 2, the simulation results exactly match the the closed form BER results for $M = 1$. Also, we find that at high signal to noise ratio (SNR), BER performance is best when the relay is closer to the source (L = 0.3).

In Fig. 3, CLT based BER results are plotted for upto $M = 8$ relays using Theorem 4.3. A comparison of the CLT approximation with the simulation results for $M = 1, 2$ shows that the CLT approximation acts as a lower bound on the BER for low SNR, but closely follows the simulations in the high SNR regions. We have considered $E_s = 0.5, E_r = \frac{0.5}{M}$ and $L_r = 0.5$. Interestingly, the plots suggest that there is a loss in the diversity order by a factor of 2. For noncoherent detection, this has actually been shown to be the case in [5]. Representing the diversity order and the corresponding number of relays as (d, M), we graphically obtain, for even number of relays, the values $(1.6, 2), (2.6, 4), (3.5, 6)$ and $(4.9, 8)$. For odd number of relays, the diversity order is obtained as $(1.3, 1), (2.3, 3), (3.7, 5)$ and $(5.0, 7)$. Thus for, the cases considered, we find that $d < \frac{M}{2} + 1, M$ even and $d < \frac{M+3}{2}, M$ odd. While there is an exact match for even number of relays, when the number of relays is odd, we find the bound on the diversity order for coherent detection to be a unit more than that obtained for the case of noncoherent detection in [5].

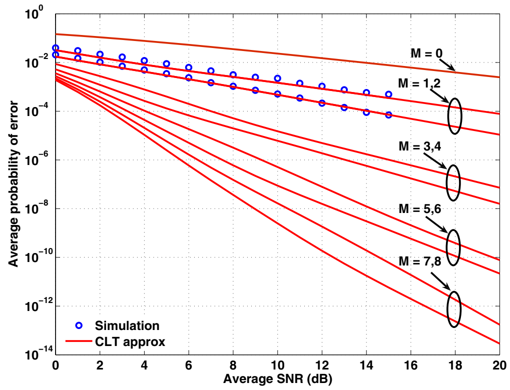

Fig. 3. Large M approximation using CLT. $M = 1 \dots 8$. DF loses half the diversity order.

VI. Conclusions

In this paper, we have found exact expressions for the BER of an ML-DF cooperative diversity system employing a single relay, for the Rayleigh fading channel. A simpler expression is obtained by using the PL approximation, again for the single relay system. We find that the BER expression for the PL combiner for a single relay system can be directly obtained by a slight approximation from the exact expression for the ML detector. In the process, we have defined conditionally Gaussian variables and found expressions for their CDF, PDF and CF. The importance of these variables is obvious from the simplicity of the derivation of the BER expressions. Further, using the central limit theorem, we obtain an approximate expression for the BER when multiple relays are employed. Using this expression, the diversity order of the system was obtained graphically. On the basis of our numerical results, we believe that there is a loss in the diversity order by a factor of two, just as as in the case of noncoherent detection [5].

Appendix

A. Proof of Theorem 3.1

The conditional CDF of Z can be expressed as

$$P(Z < z|A) = \begin{cases} 1 - Q\left(\frac{-aA+z}{\sqrt{bA}}\right) & z \geq 0 \\ Q\left(\frac{aA-z}{\sqrt{bA}}\right) & z < 0 \end{cases}. \quad (32)$$

Unconditioning (32) with respect to A, we have

$$F_Z(z) = \begin{cases} 1 - c\int_0^\infty Q\left(\frac{-ax+z}{\sqrt{bx}}\right) e^{-cx} dx & z \geq 0 \\ c\int_0^\infty Q\left(\frac{ax-z}{\sqrt{bx}}\right) e^{-cx} & z < 0 \end{cases}. \quad (33)$$

Now, applying Lemma 3.1, we obtain (14). Differentiating $F_Z(z)$ in (14) results in the expression for $p_Z(z)$ in (15). The CF of Z can be expressed using (15) as

$$\begin{aligned}\phi_Z(t) &= E[e^{jtZ}] = \int_{-\infty}^{\infty} p_Z(x) e^{jtx} dx \\ &= g\int_0^\infty e^{-xh(-1)} e^{jtx} dx + g\int_{-\infty}^0 e^{xh(1)} e^{jtx} dx.\end{aligned}$$

After evaluating the above integrals and simplifying, we obtain (16).

B. Proof of Theorem 3.2

Since $V = f(Z)$, the CDF of V can be expressed as

$$\begin{aligned}F_V(v) &= P(V < v) = P\left(f(Z) < v\right) \\ &= P\left(e^Z < s(-v)\right). \end{aligned} \quad (34)$$

Since $e^Z > 0$,

$$\begin{aligned}F_V(v) &= P\left(Z < \ln(s(-v))\right) \\ &= \begin{cases} 1 & v > \ln\frac{1}{\delta} \\ F_Z(\ln(s(-v))) & \ln\delta < v < \ln\frac{1}{\delta} \\ 0 & v < \ln\delta \end{cases}\end{aligned} \quad (35)$$

Using the expression for the CDF of Z from Theorem 3.1 in (35) and noting that $\ln(s(v))$ is odd, we obtain (18). The characteristic function of V is given by

$$\phi_V(t) = E[e^{jtV}] = \int_{-\infty}^{\infty} e^{jtf(z)} p_Z(z) dz. \quad (36)$$

Now, using the PL approximation from Lemma 2.1, we obtain

$$\begin{aligned}\phi_V(t) = &\int_{-\infty}^{\ln\delta} e^{jt\ln\delta} p_Z(z) dz + \int_{\ln\delta}^{0} e^{jtz} p_Z(z) dz \\ &\int_0^{\ln\frac{1}{\delta}} e^{jtz} p_Z(z) dz + \int_{\ln\frac{1}{\delta}}^{\infty} e^{jt\ln\frac{1}{\delta}} p_Z(z) dz.\end{aligned} \quad (37)$$

Substituting the expression for $p_Z(z)$ from (15) in the above and simplifying, we obtain (19).

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