Finding independent sets in hereditary graph classes

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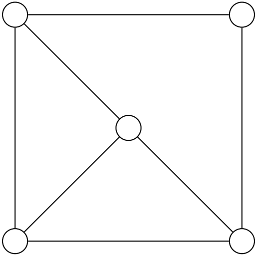
Maximum Independent Set (MIS)

Problem: Given a graph



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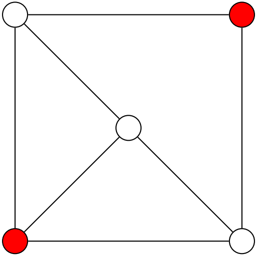

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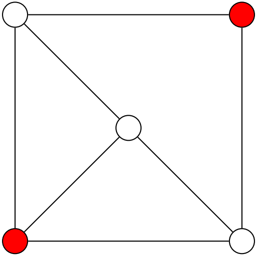

and an integer k: Is there an independent set of size at least k?

- classical **NP-complete** problem
- $W[1]$-complete (no algorithm running in $f(k)n^{O(1)}$ time)
- no $n^{1-\varepsilon}$-approximation unless $P = NP$

What about on restricted graphs classes?

Maximum Independent Set (MIS) in restricted graph classes

Among others:

- polynomial in bipartite graphs (Kőnig's theorem)
- polynomial in chordal graphs (simplicial decomposition)
- polynomial in perfect graphs (ellipsoid method)
- polynomial in P_6-free graphs [Grzesik et al, SODA 19]
- NP-complete in graphs with maximum degree 3
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General question:

Given a fixed graph H, can we solve Maximum Independent Set more efficiently in H-free graphs[a]?

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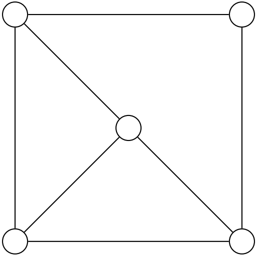
"more efficiently": polynomial? approximation? FPT?

NP-hard cases [Poljak, 1974, Alekseev '82]



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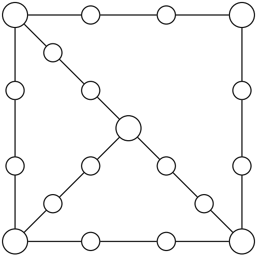

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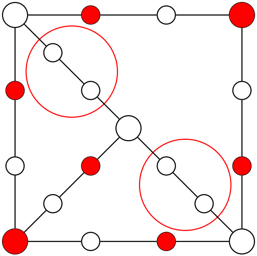

Subdivide every edge twice $\Rightarrow \alpha(G') = \alpha(G) + |E(G)|$

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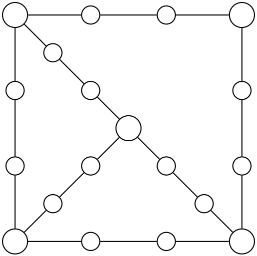

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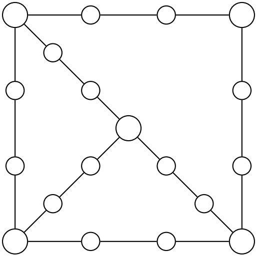

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Maximum Independent Set remains NP-hard in H-free graphs except if H is...

P/NP-complete status of MIS on H-free graphs

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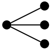
- NP-complete, if H is **<u>not</u>** a path or a subdivided claw (claw = )
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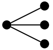
- NP-complete, if H is **<u>not</u>** a path or a subdivided claw (claw = )
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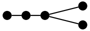
- in P, if H is a claw with one edge subdivided once ( a.k.a. the fork)

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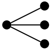
- NP-complete, if H is **not** a path or a subdivided claw (claw = )
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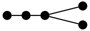
- in P, if H is a claw with one edge subdivided once ( a.k.a. the fork)
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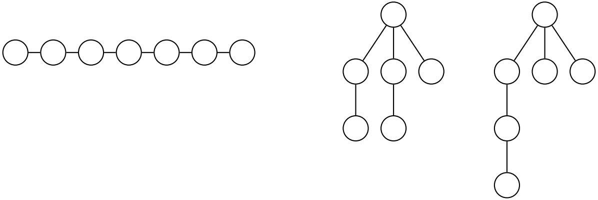
Minimal open cases:



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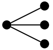
- NP-complete, if H is **<u>not</u>** a path or a subdivided claw (claw = )
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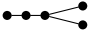
- in P, if H is a claw with one edge subdivided once ( a.k.a. the fork)
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Here, we focus on NP-complete cases

When a problem is NP-complete:

Parameterized algorithms

A problem is **Fixed-Parameter Tractable** (FPT) if it can be solved in time $f(k)n^{O(1)}$, where

- n is the size of an instance
- k is a parameter (here: size of the independent set we are looking for)
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Unless FPT $\neq$ W[1], Maximum Independent Set (in general graphs) is **not** FPT

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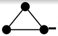
Toy example: Independent Set is FPT in -free graphs:

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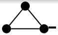
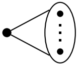
Toy example: Independent Set is FPT in -free graphs:

- case 1: there is a vertex of degree $\geq k$: 

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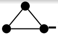
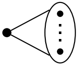
Toy example: Independent Set is FPT in -free graphs:

- case 1: there is a vertex of degree $\geq k$:  

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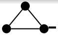
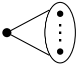
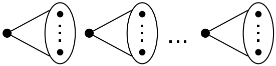
Toy example: Independent Set is FPT in -free graphs:

- case 1: there is a vertex of degree $\geq k$:  
- case 2 : $d(v) < k$ for all $v \in V$:



  - either we construct an independent set of size k
  - or the graph has at most $k(k-1) = O(k^2)$ vertices $\Rightarrow$ brute-force

More generally:

Ramsey's theorem

Given $r, k > 1$, there exists an integer $Ram(r, k)$ such that any graph G with at least $Ram(r, k)$ vertices must contain either:

- a clique of size r, or
- an independent set of size k

Example: $Ram(3, 3) = 6$, $Ram(4, 4) = 18$

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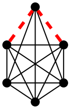
Maximum Independent Set is FPT in K_r-free graphs for every $r \geq 1$

(we say it admits a **kernel** with $O(k^{r-1})$ vertices)

Now : what if the forbidden graph H is "almost" a clique?



Let's call it K_r^{-2}

$\rightarrow$ can't use Ramsey here...

Erdős-Hajnal Conjecture

H satisfies the **Erdős-Hajnal** property if there is $0 < \varepsilon_H \leq 1$ such that for any H-free graph G on n vertices, either:

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Graphs known to satisfy the EH-property:

- K_r (simple induction)
- any graph on four vertices
- graphs that can be constructed from them by "substitution operation"

Open for many graphs, in particular:

- C_5: cycle on five vertices
- P_5: path on five vertices

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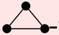
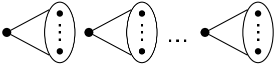
K_r satisfies the EH-property (e.g. -free graphs have $\alpha(G) \geq \sqrt{n}$)

by induction on r:

- if there is a vertex v of degree $\geq n^{\frac{r-2}{r-1}}$:
  - $N(v)$ is K_{r-1}-free (by induction)

    $\Rightarrow$ there is an independent set of size $\geq \left(n^{\frac{r-2}{r-1}}\right)^{\frac{1}{r-2}} = n^{\frac{1}{r-1}}$
- otherwise:



We construct an independent set of size $\geq \frac{n}{n^{\frac{r-2}{r-1}}} = n^{\frac{1}{r-1}}$

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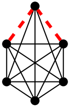
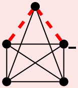
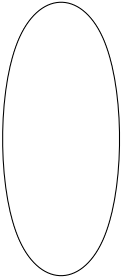
K_r^{-2} satisfies the Erdős-Hajnal property, and an independent set or a clique of size $n^{\frac{1}{r-1}}$ **can be found in polynomial time** (simple induction)



Example: $O(k^4)$ kernel in -free graphs

- invoke Erdős-Hajnal algorithm:
  - either large independent set $\rightarrow$ done
  - or large clique $\geq n^{1/4}$



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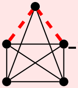
Example: $O(k^4)$ kernel in  -free graphs

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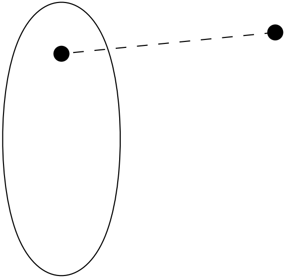
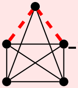


Example: $O(k^4)$ kernel in -free graphs

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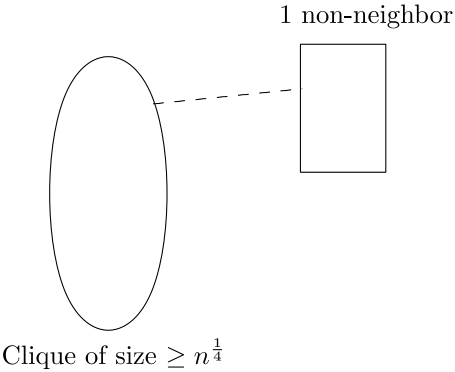
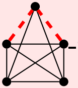


Example: $O(k^4)$ kernel in -free graphs

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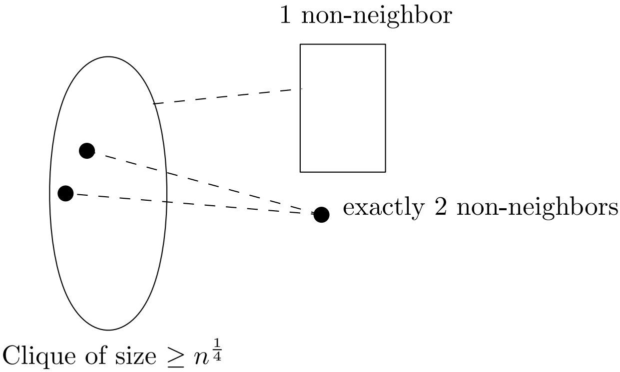
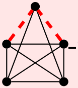


Example: $O(k^4)$ kernel in  -free graphs

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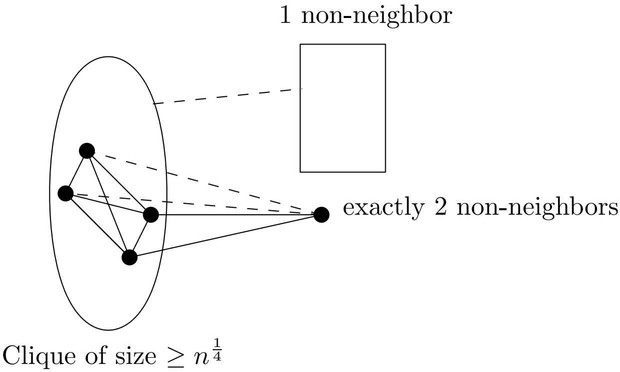
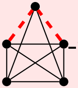


Example: $O(k^4)$ kernel in -free graphs

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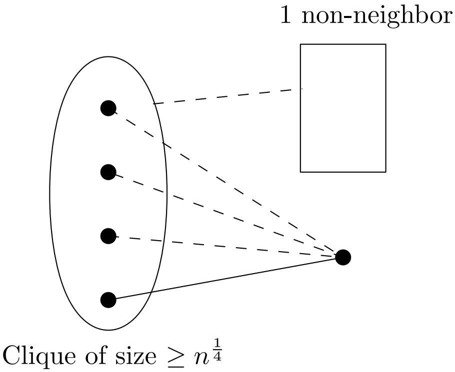
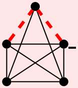


Example: $O(k^4)$ kernel in -free graphs

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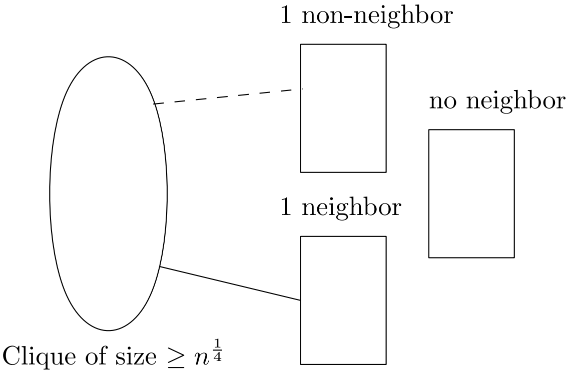
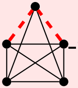


Example: $O(k^4)$ kernel in -free graphs

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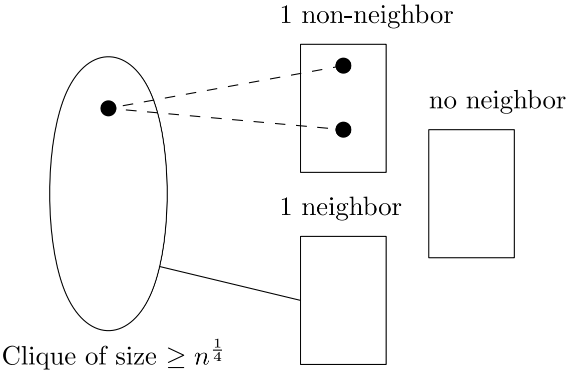
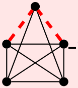


Example: $O(k^4)$ kernel in -free graphs

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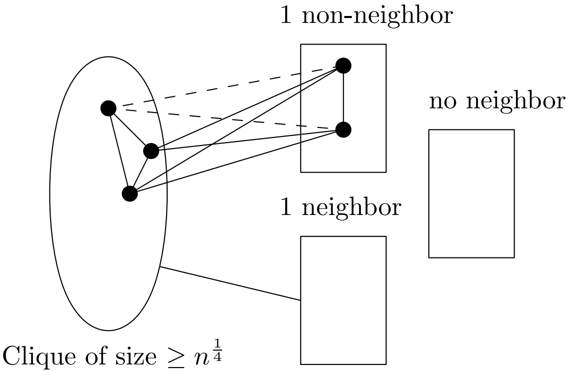
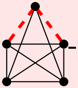


Example: $O(k^4)$ kernel in  -free graphs

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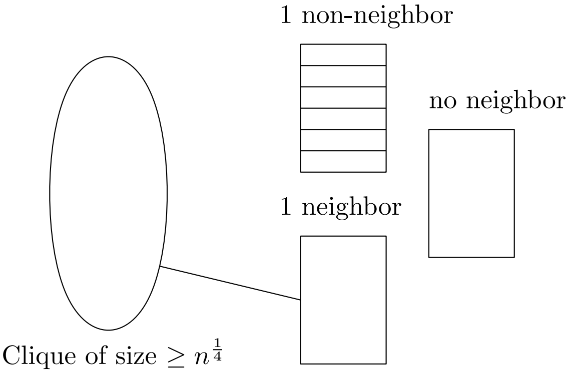
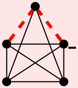


Example: $O(k^4)$ kernel in -free graphs

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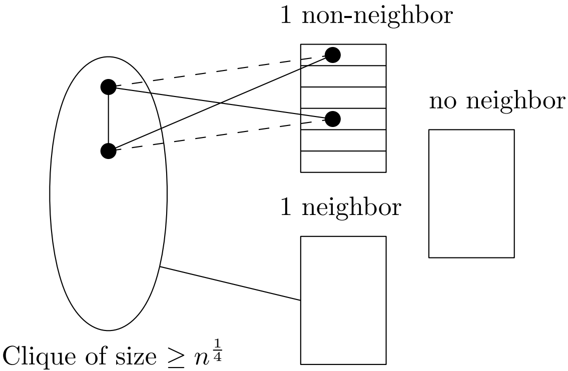
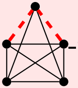


Example: $O(k^4)$ kernel in  -free graphs

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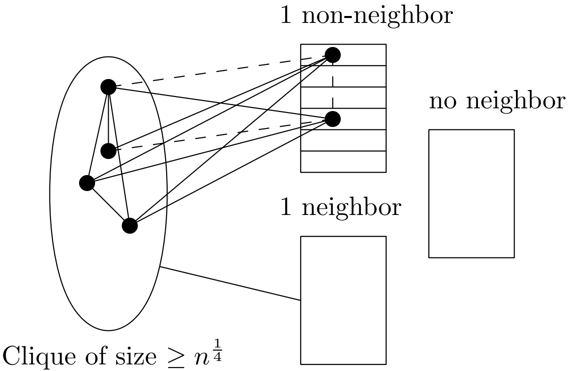
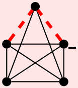


Example: $O(k^4)$ kernel in  -free graphs

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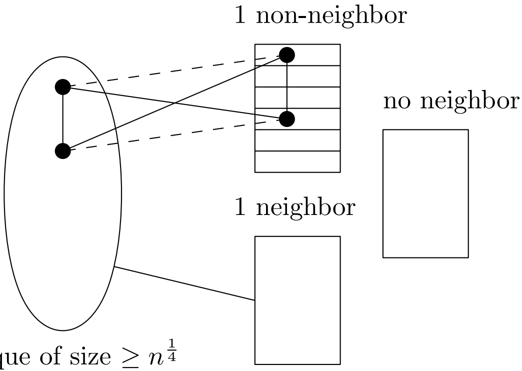
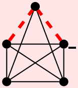


Example: $O(k^4)$ kernel in  -free graphs

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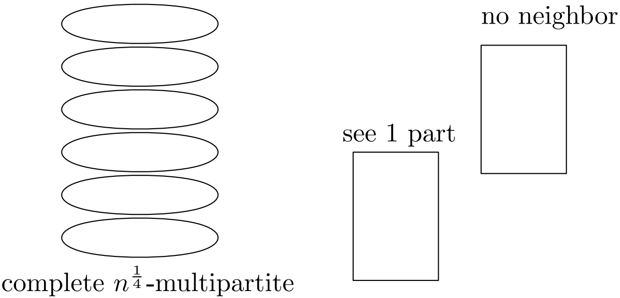
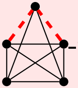


Example: $O(k^4)$ kernel in -free graphs

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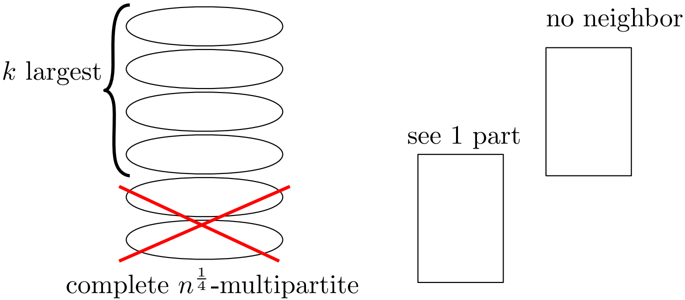
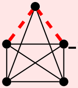


Example: $O(k^4)$ kernel in -free graphs

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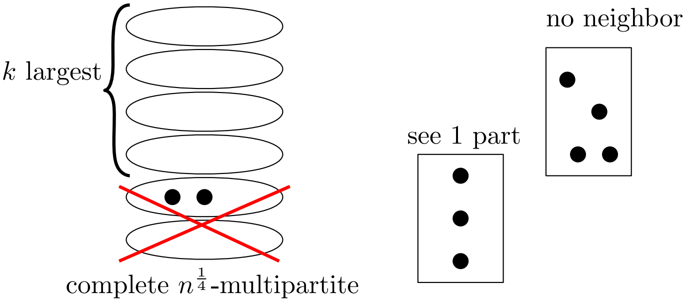
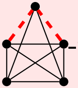


Example: $O(k^4)$ kernel in -free graphs

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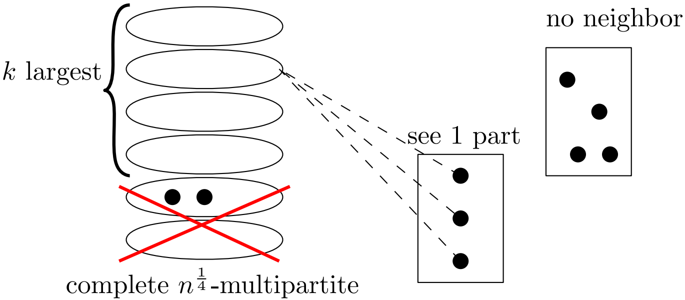
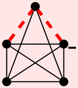


Example: $O(k^4)$ kernel in -free graphs

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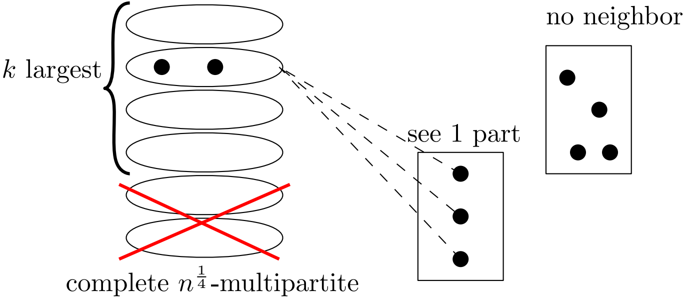
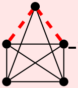


Example: $O(k^4)$ kernel in -free graphs

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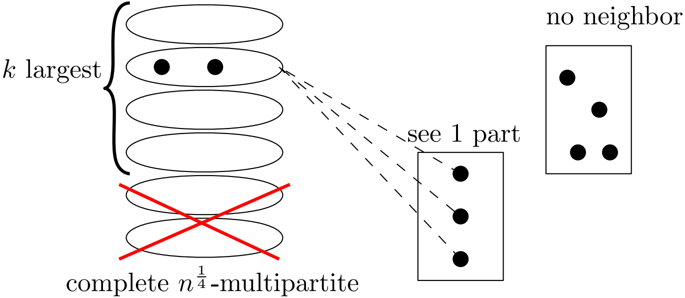
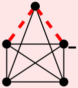


Example: $O(k^4)$ kernel in -free graphs

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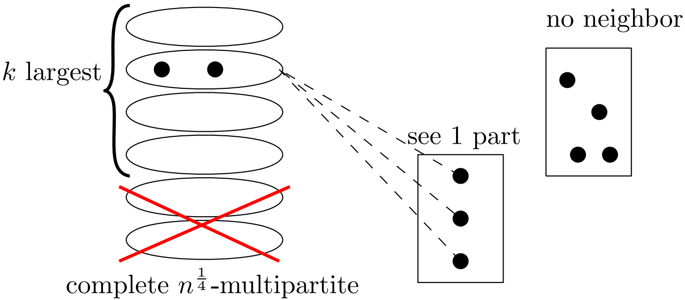

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FPT	**not FPT**
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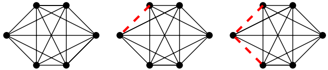

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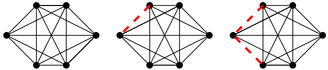

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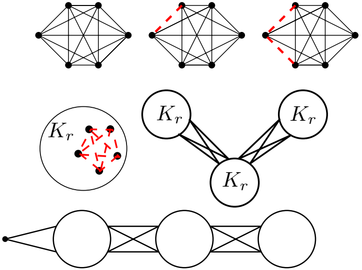

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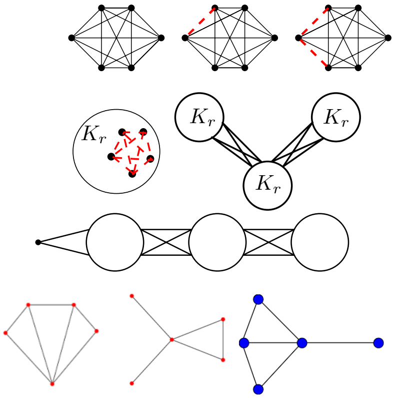

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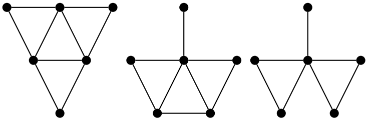

Open questions

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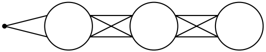
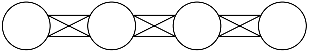
- generalizing  to 

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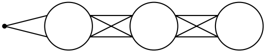
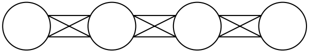
- generalizing  to 
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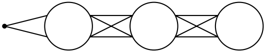
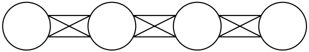
- generalizing  to 
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