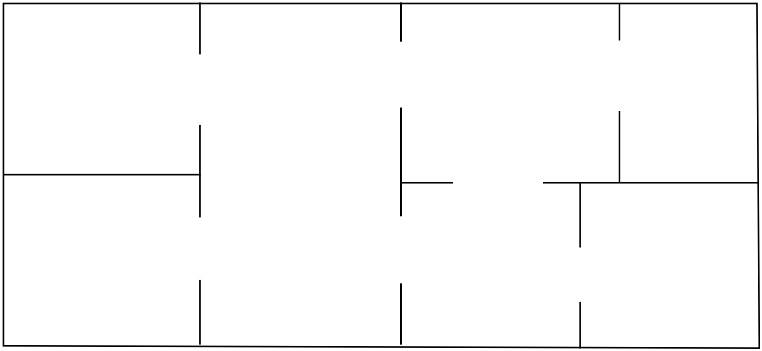
Identification problems on graphs

selected topics

Florent Foucaud
Université de Bordeaux

CALDAM pre-conference school, February 2020

Locating a burglar



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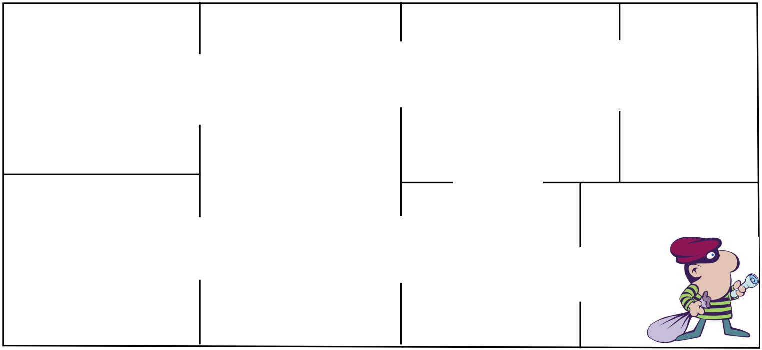

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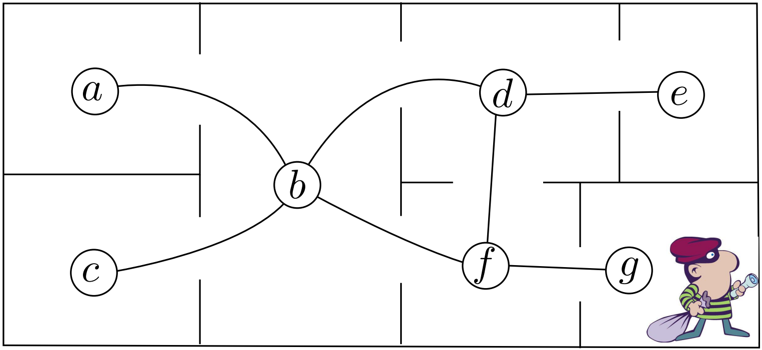

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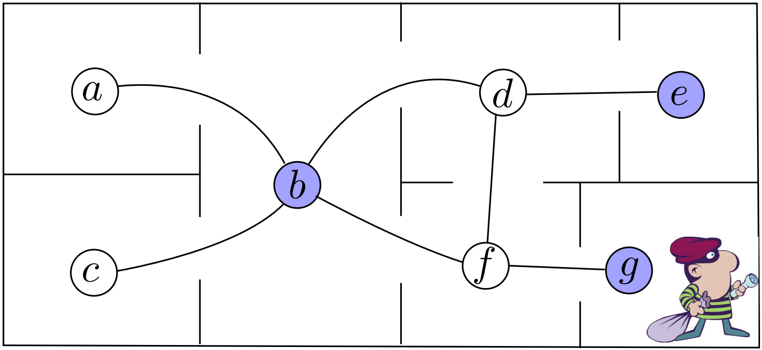

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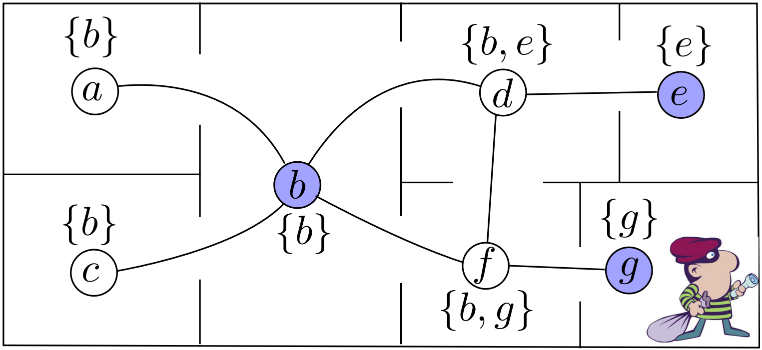

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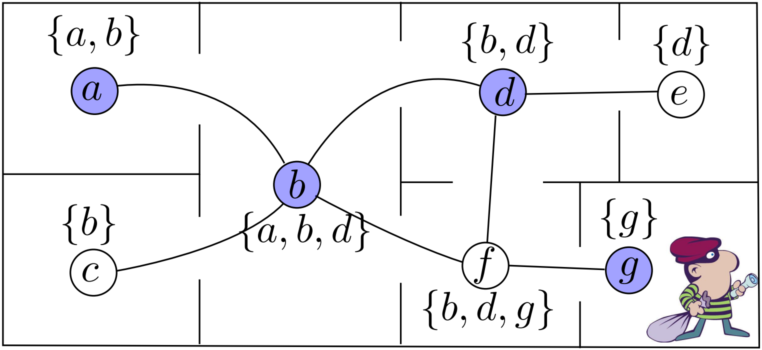
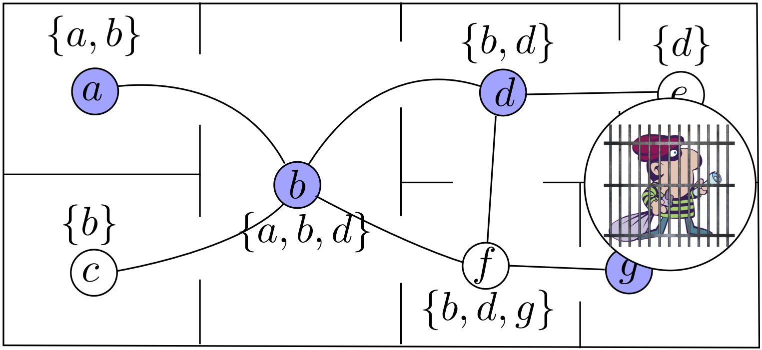
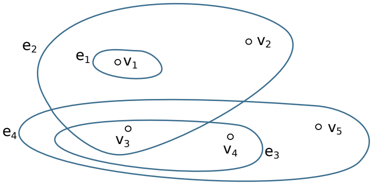



{a, b}
a
{b, d}
d
{d}
e
b
{a, b, d}
{b}
c
f
g
{b, d, g}

Separating sets in hypergraphs

Definition - Separating set (Rényi, 1961)

Hypergraph $(X,\mathscr{E})$. A separating set is a subset $C \subseteq X$ such that each edge $e \in \mathscr{E}$ contains a distinct subset of C.



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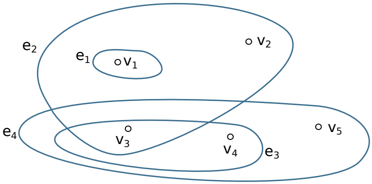

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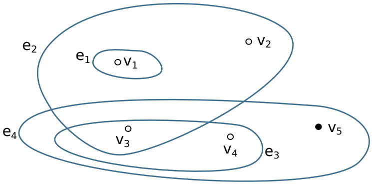

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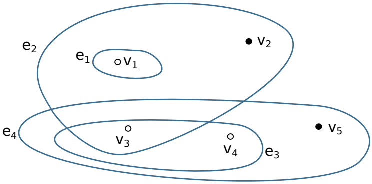

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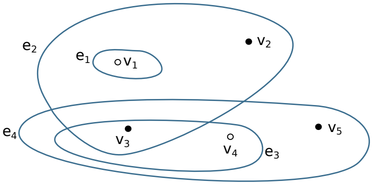

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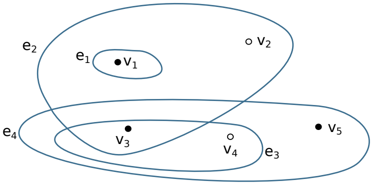

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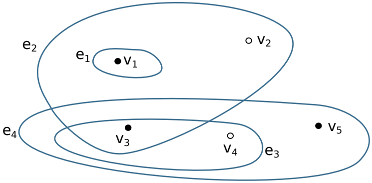

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Also known as Separating system, Distinguishing set, Test cover, Distinguishing transversal, Discriminating code...

Proposition

For a hypergraph $(X, \mathscr{E})$, a separating set C has size at least $\log_2(|\mathscr{E}|)$.

Proof: Must assign to each edge, a distinct subset of C: $|\mathscr{E}| \leq 2^{|C|}$. $\square$

General bounds, Bondy's theorem

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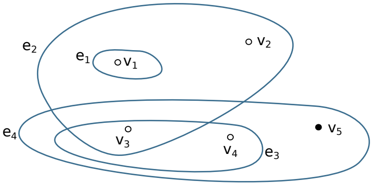
Which are the "problematic" vertices?



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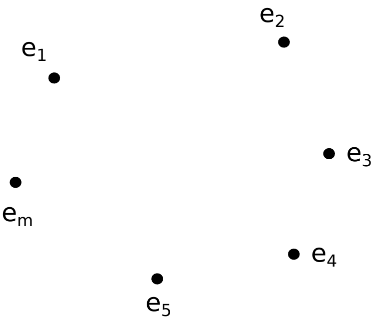
Build graph G on vertex set $V(G) = \mathscr{E}$.



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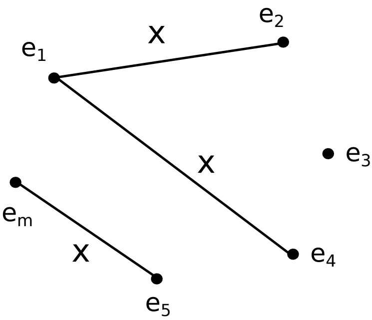

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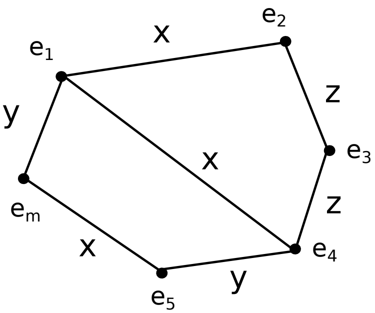

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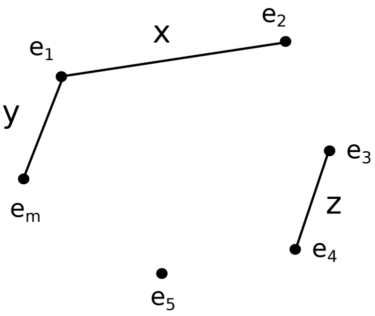

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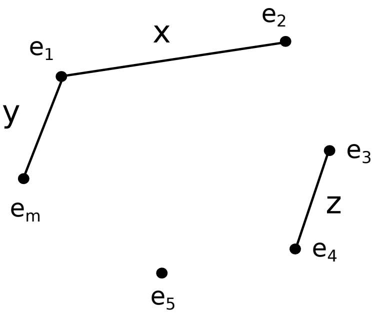

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So, there are at most $|\mathscr{E}| - 1$ "problematic" vertices. $\rightarrow$ Find one "non-problematic vertex" and omit it. □

Some example problems

Special graph-based cases of separating sets in hypergraphs:

- **identifying codes**
- identifying open codes
- path/cycle identifying covers
- separating path systems

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Distance-based identification:

- **resolving sets** (metric dimension)
- centroidal locating sets
- tracking paths problem

Identifying codes in graphs

G: undirected graph
$N[u]$: set of vertices v s.t. $d(u,v) \leq 1$

Definition - Identifying code (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of $V(G)$ such that:

- C is a **dominating set**: $\forall u \in V(G)$, $N[u] \cap C \neq \emptyset$, and
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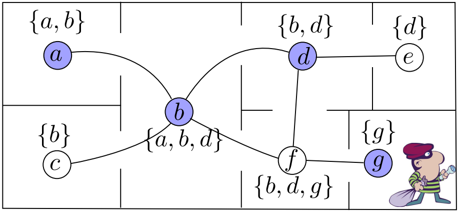
$ID(G)$: identifying code number of G,
minimum size of an identifying code in G



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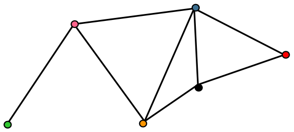

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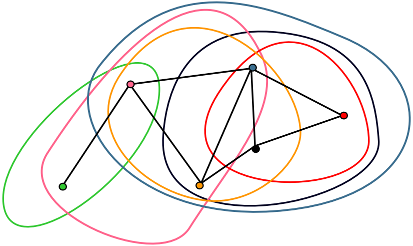

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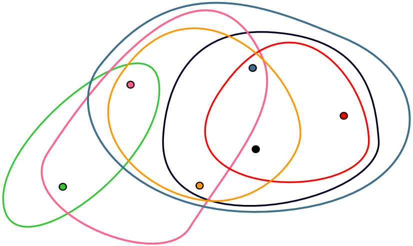

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Domination number: $DOM(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Examples: paths

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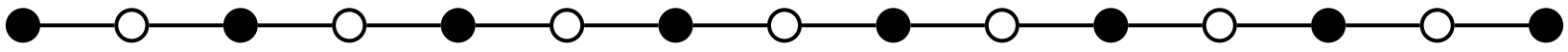
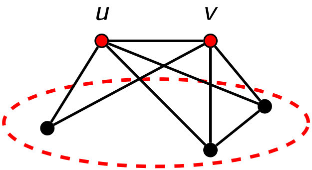


Remark

Not all graphs have an identifying code!

Closed twins = pair u, v such that $N[u] = N[v]$.



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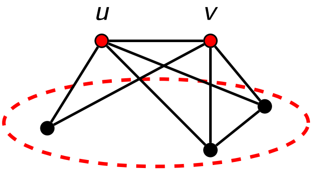

Proposition

A graph is **identifiable** if and only if it is **closed twin-free** (i.e. has no twins).

Bounds on $ID(G)$

n: number of vertices

Proposition

G identifiable graph on n vertices: $\lceil \log_2(n+1) \rceil \leq ID(G)$.

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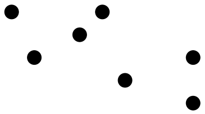
$ID(G) = n \Leftrightarrow G$ has no edges



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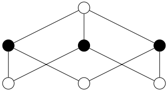

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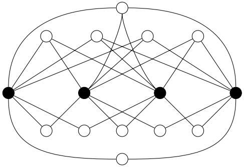
$ID(G) = \log_2(n+1)$



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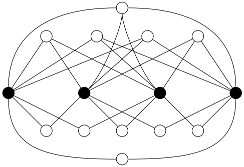

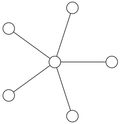

Further examples

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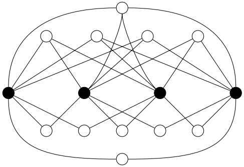

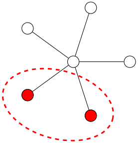

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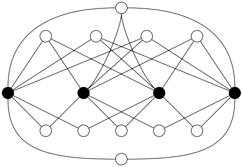

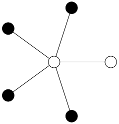

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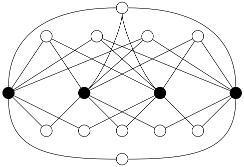

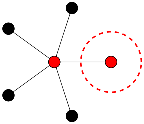

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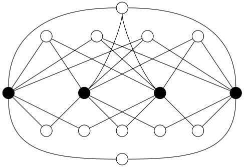

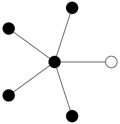

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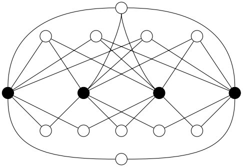

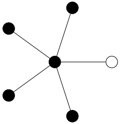

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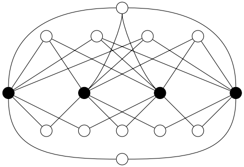

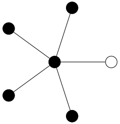

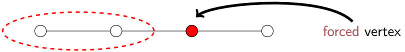

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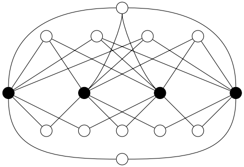
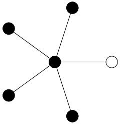
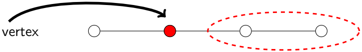




forced vertex



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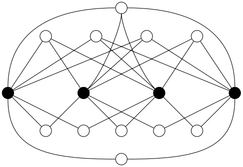

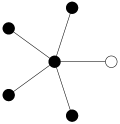

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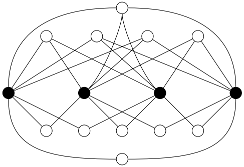
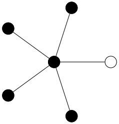




Theorem (Bertrand, 2005 / Gravier, Moncel, 2007 / Skaggs, 2007)

G identifiable graph on n vertices with at least one edge:

$$ID(G) \leq n-1$$

Question

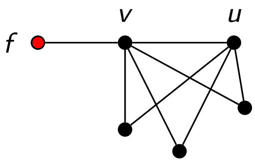
What are the graphs G with n vertices and $ID(G) = n-1$?

Forced vertices

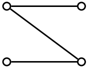
u, v such that $N[v] \ominus N[u] = \{f\}$:

f belongs to **any identifying code**

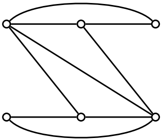
$\rightarrow$ f forced by u, v.



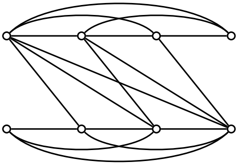
Special path powers: $A_k = P_{2k}^{k-1}$



$A_2 = P_4$



$A_3 = P_6^2$



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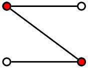

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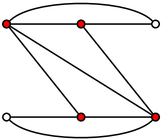

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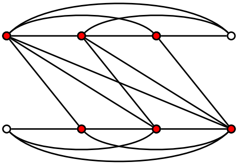

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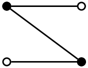

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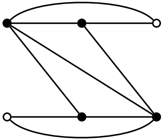

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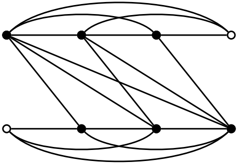

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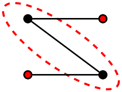

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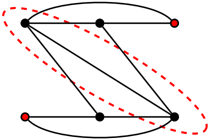

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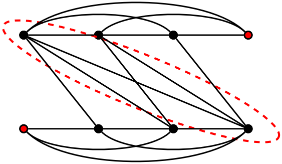

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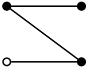

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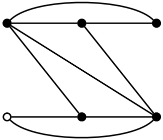

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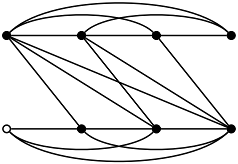
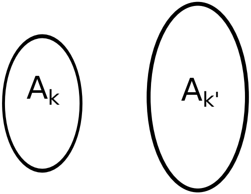


$A_4 = P_8^3$

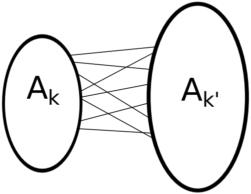
Proposition

$$ID(A_k) = n - 1$$



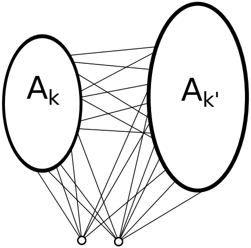
Two graphs A_k and $A_{k'}$

Constructions using joins



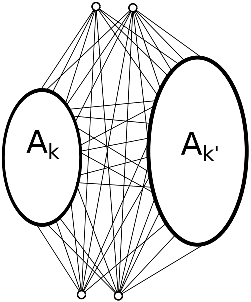
Join: add all edges between them

Constructions using joins

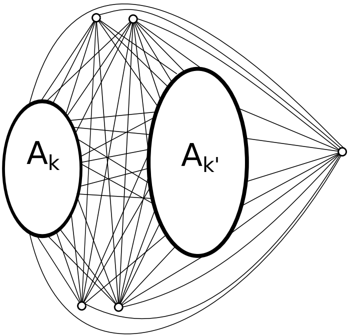


Join the new graph to two non-adjacent vertices ($\overline{K_2}$)

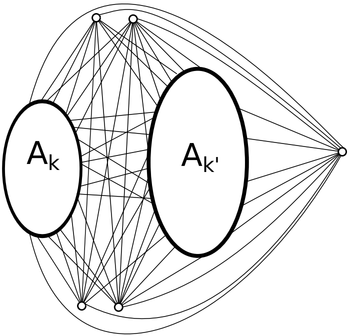
Constructions using joins



Join the new graph to two non-adjacent vertices, again



Finally, add a **universal vertex**



Finally, add a **universal vertex**

Proposition

At each step, the constructed graph has $ID = n - 1$

A characterization

(1) stars

(2) $A_k = P_{2k}^{k-1}$

(3) joins between 0 or more members of (2) and 0 or more copies of $\overline{K_2}$

(4) (2) or (3) with a universal vertex

Theorem (F., Guerrini, Kovše, Naserasr, Parreau, Valicov, 2011)

G connected identifiable graph, n vertices:

$$ID(G) = n-1 \Leftrightarrow G \in (1), (2), (3) \text{ or } (4)$$

Lower bounds

Proposition

G identifiable graph on n vertices: $\lceil \log_2(n+1) \rceil \leq ID(G)$.

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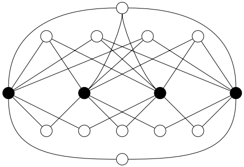
Tight example ($k = 4$):



Proposition

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Theorem (Rall & Slater, 1980's)

G planar graph, order n, $ID(G) = k$. Then $n \leq 7k - 10 \rightarrow ID(G) \geq \frac{n+10}{7}$.

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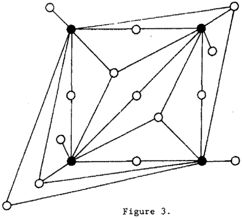
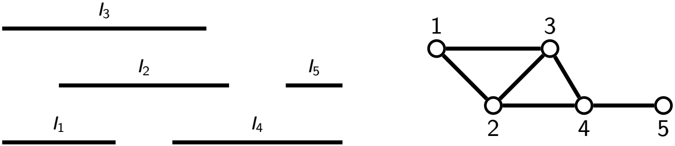
Tight examples: 

Figure 3.

Definition - Interval graph

Intersection graph of intervals of the real line.



Lower bound for interval graphs

Theorem (F., Mertzios, Naserasr, Parreau, Valicov, 2017)

G interval graph of order n, $ID(G) = k$.

Then $n \leq \frac{k(k+1)}{2}$, i.e. $ID(G) = \Omega(\sqrt{n})$.

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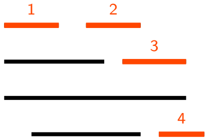

- Identifying code D of size k.
- Define zones using the right points of intervals in D.

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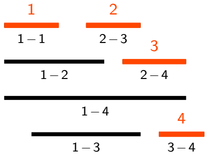

- Identifying code D of size k.
- Define zones using the right points of intervals in D.
- Each vertex intersects a consecutive set of intervals of D when ordered by left points.

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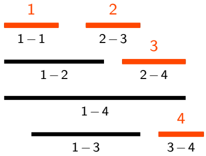

- Identifying code D of size k.
- Define zones using the right points of intervals in D.
- Each vertex intersects a consecutive set of intervals of D when ordered by left points.

$\rightarrow n \leq \sum_{i=1}^{k}(k-i) = \frac{k(k+1)}{2}$.

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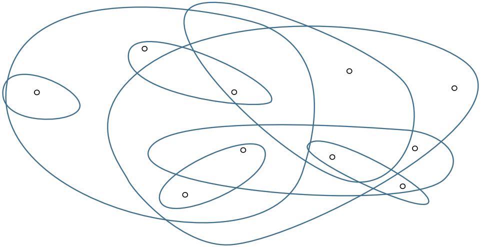
Tight:

Measure of intersection complexity of sets in a hypergraph $(X, \mathscr{E})$
(initial motivation: machine learning, 1971)

A set $S \subseteq X$ is shattered:
for every subset $S' \subseteq S$, there is an edge e with $e \cap S = S'$.



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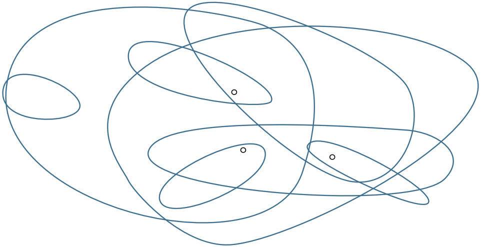

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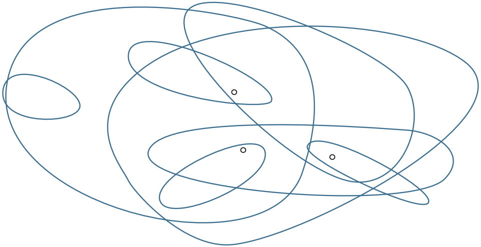
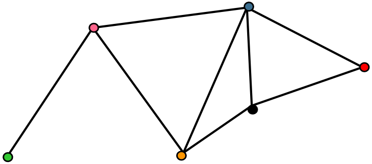


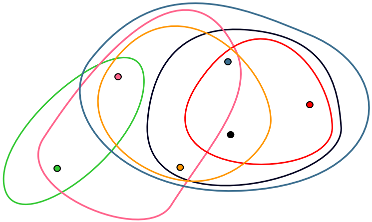
V-C dimension of H: maximum size of a shattered set in H

Typically bounded for geometric hypergraphs:

Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its closed neighbourhood hypergraph





Vapnik-Červonenkis dimension - graphs

V-C dimension of a graph: V-C dimension of its closed neighbourhood hypergraph

Typically bounded for geometric intersection graphs:

$\rightarrow$ interval graphs ($d = 2$), C_4-free graphs ($d = 2$), line graphs ($d = 4$), permutation graphs ($d = 3$), unit disk graphs ($d = 3$), planar graphs ($d = 4$)...

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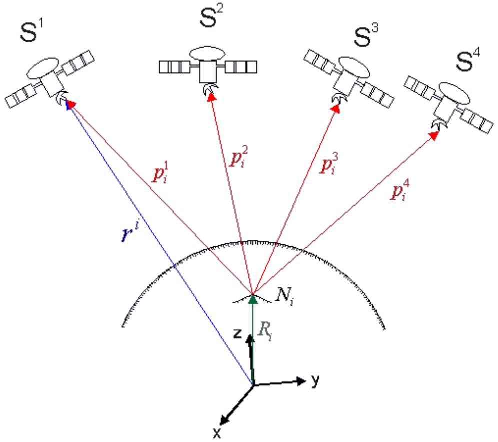
Corollary

G graph of order n, $ID(G) = k$, V-C dimension $\leq d$. Then $n = O(k^d)$.

Metric dimension

Determination of Position in 3D euclidean space

GPS/GLONASS/Galileo/Beidou/IRNSS:
need to know the exact position of 4 satellites + distance to them



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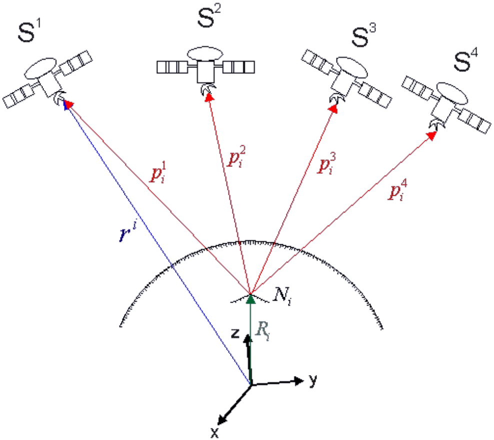

Question

Does the "GPS" approach also work in undirected unweighted graphs?

Metric dimension

Now, $w \in V(G)$ distinguishes $\{u,v\}$ if $dist(w,u) \neq dist(w,v)$

Definition - Resolving set (Slater, 1975 - Harary & Melter, 1976)

$R \subseteq V(G)$ resolving set of G:

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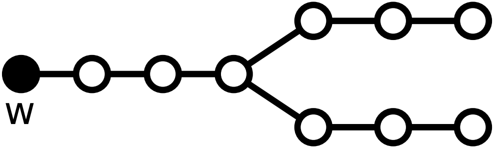

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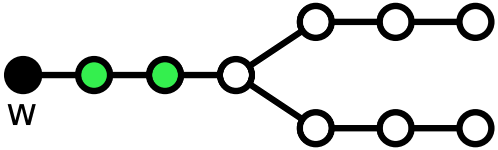

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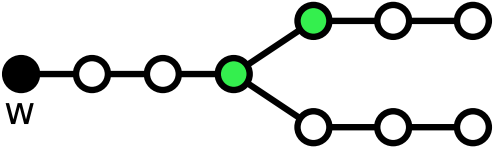

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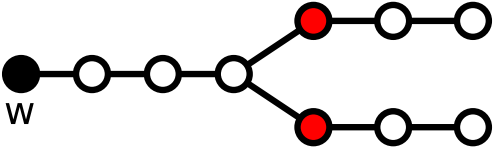

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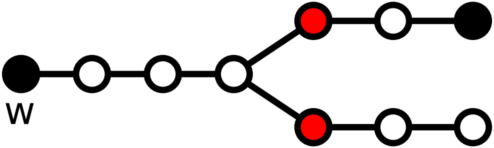

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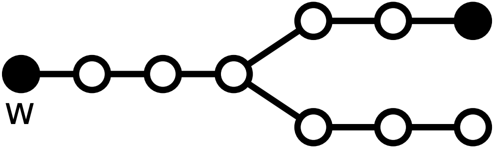

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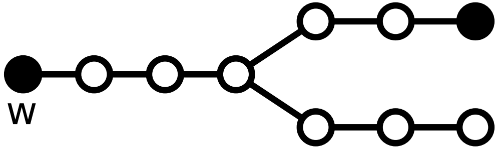

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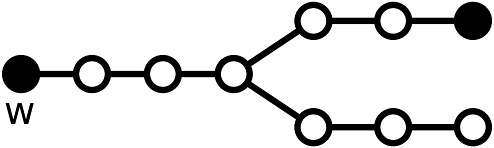

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$MD(G) = 1 \Leftrightarrow G$ is a path

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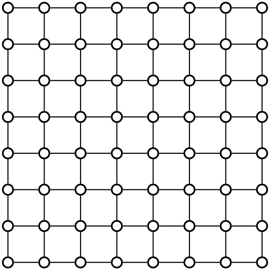

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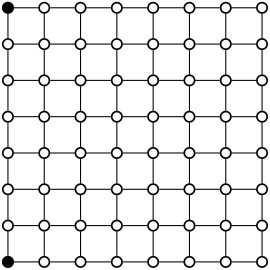
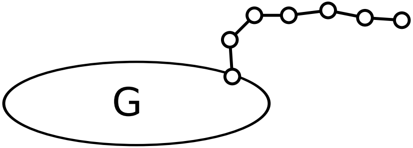


Proposition

For any square grid G, $MD(G) = 2$.

Leg: path with all inner-vertices of degree 2, endpoints of degree ≥ 3 and 1.



Trees

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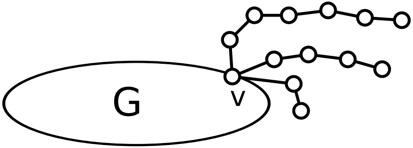

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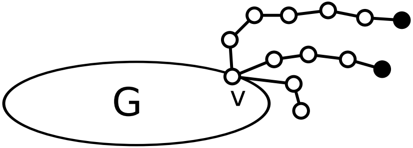

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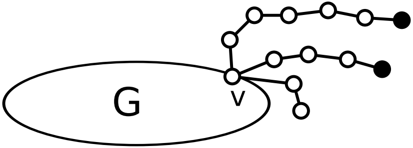

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Theorem (Slater 1975)

For any tree, the simple leg rule produces an optimal resolving set.

Example of path: no bound $n \leq f(MD(G))$ possible.

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Theorem (Khuller, Raghavachari & Rosenfeld, 2002)

G of order n, diameter D, $MD(G) = k$. Then $n \leq D^k + k$.

(diameter: maximum distance between two vertices)

Bounds with diameter

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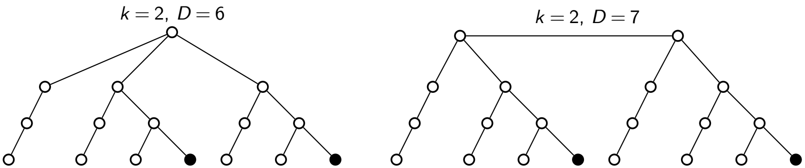
$\rightarrow$ Proofs are similar as for identifying codes.

Theorem (Beaudou, Dankelmann, F., Henning, Mary, Parreau, 2018)

T a tree with diameter D and $MD(T) = k$, then

$$n \leq \begin{cases} \frac{1}{8}(kD+4)(D+2) & \text{if } D \text{ even,} \\ \frac{1}{8}(kD-k+8)(D+1) & \text{if } D \text{ odd.} \end{cases} = \Theta(kD^2)$$

Bounds are tight.



Planar graphs

Using the concept of distance-VC-dimension:

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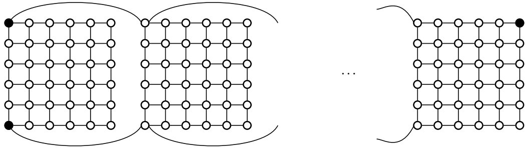
Tight? Example with $k = 3$ and $n = \Theta(D^3)$:



THANKS FOR YOUR ATTENTION