Some Computational Aspects of Knot Theory

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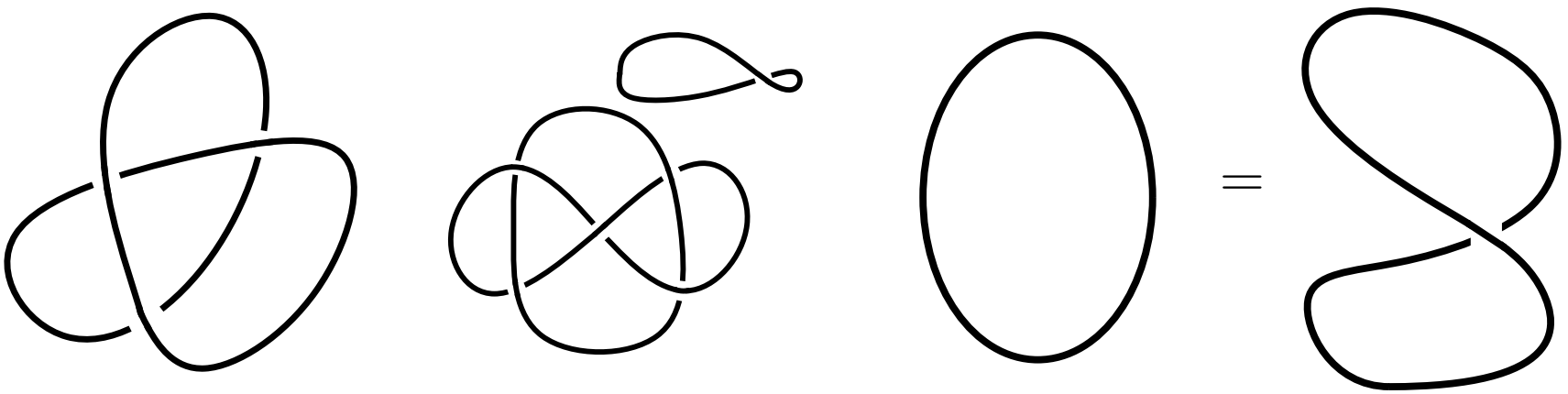
CALDAM pre-conference school 2020

What is a “mathematical knot” ?

Knots and Diagrams



- Knot: embedding of $S^1 \to \mathbb{R}^3$.
- Link: embedding of $S^1 \times \ldots \times S^1 \to \mathbb{R}^3$.
- Knot diagram: projection of the knot in the plane.

Two knots K_1 and K_2 are **equivalent** if "one can continuously deform the space" to turn K_1 into K_2.

Knot theory

Knot theory consists of designing techniques to tell non-equivalent knots apart. $\longrightarrow$ how can a computer help?

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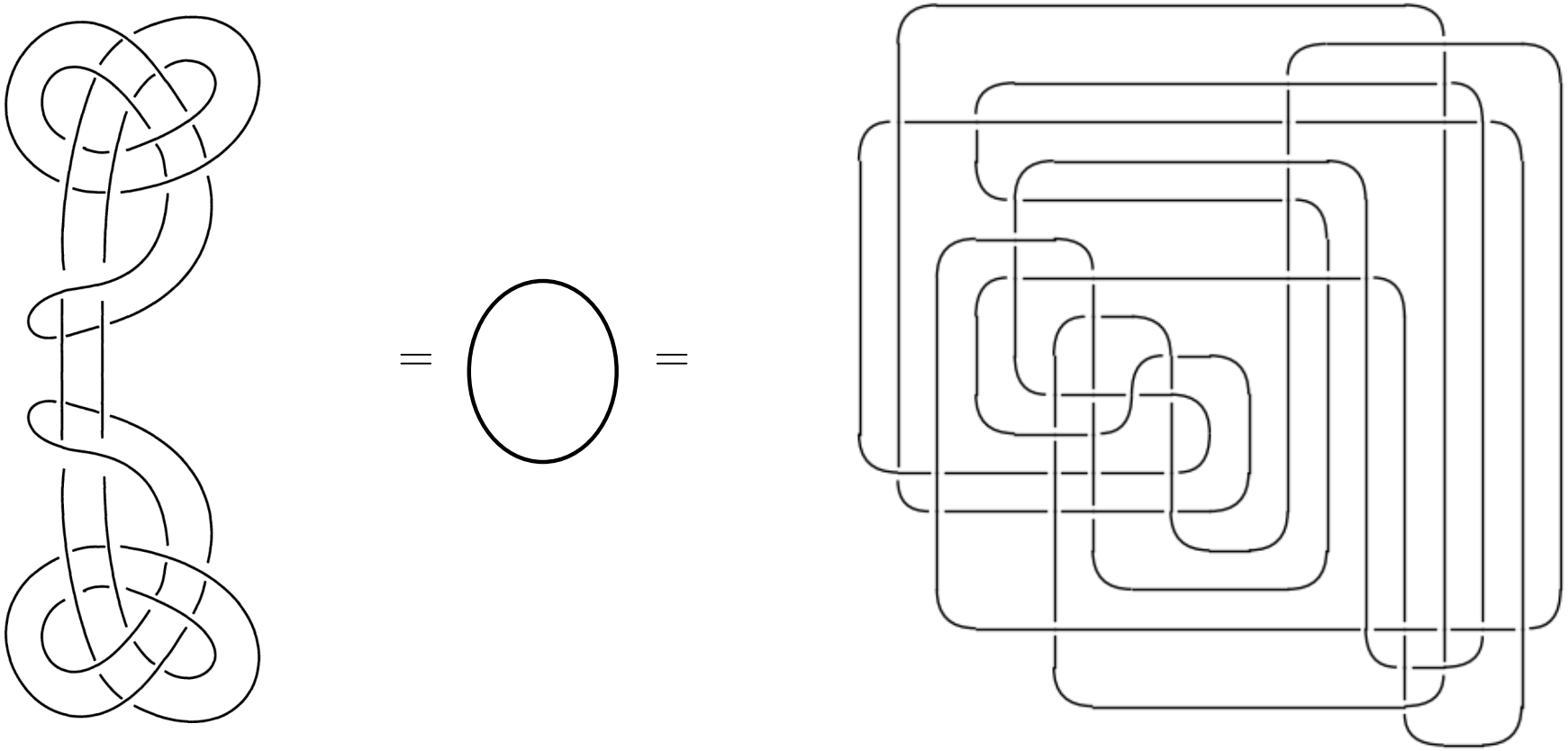
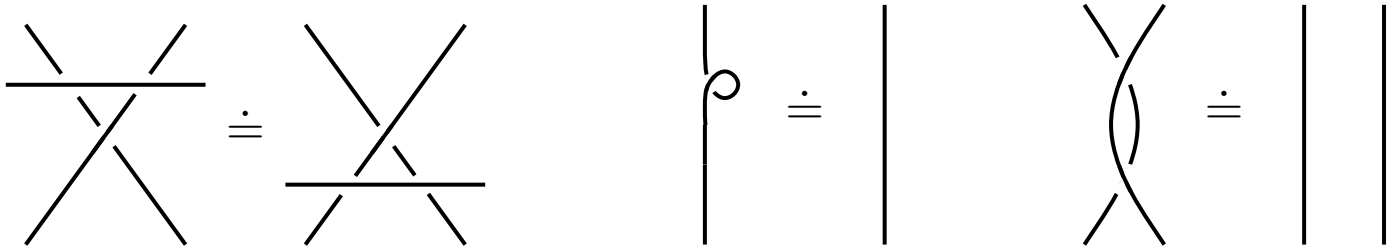
Ex: recognising the <u>trivial knot</u>:

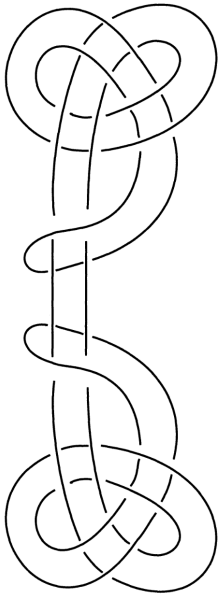


The “untangling” approach

Theorem (Reidemeister 1927)

Any two diagrams of the same *knot/link can be connected by a sequence of the following three* $\boxed{\textit{local}}$ $\boxed{\textit{combinatorial}}$ *moves, called Reidemeister moves:*





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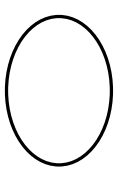

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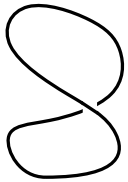

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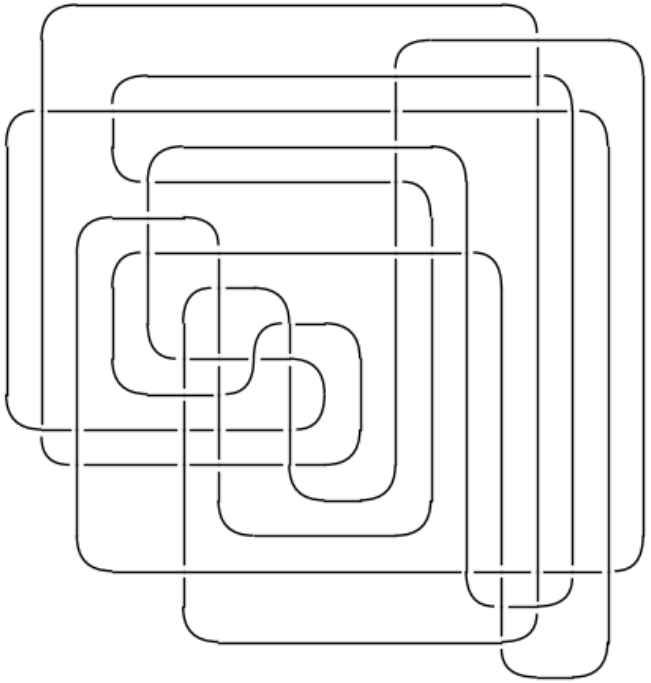
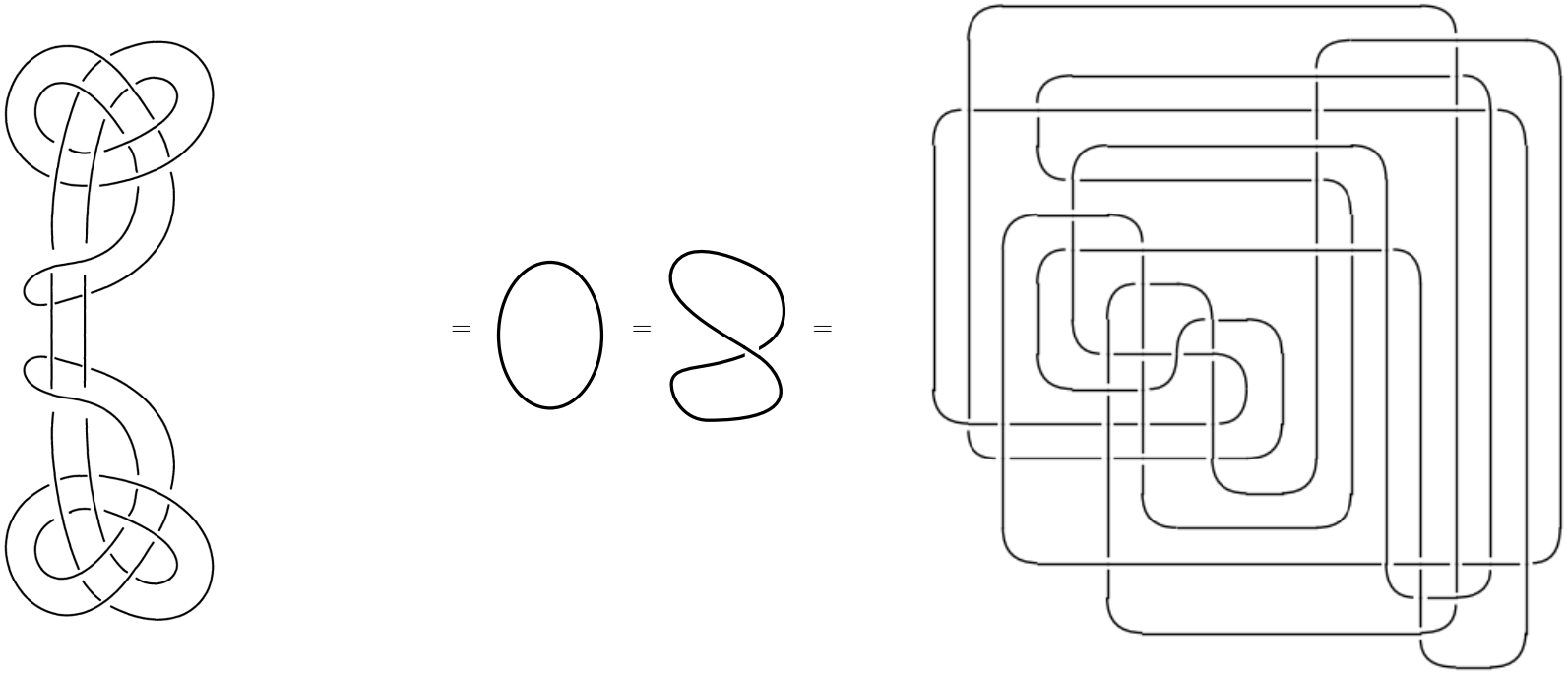
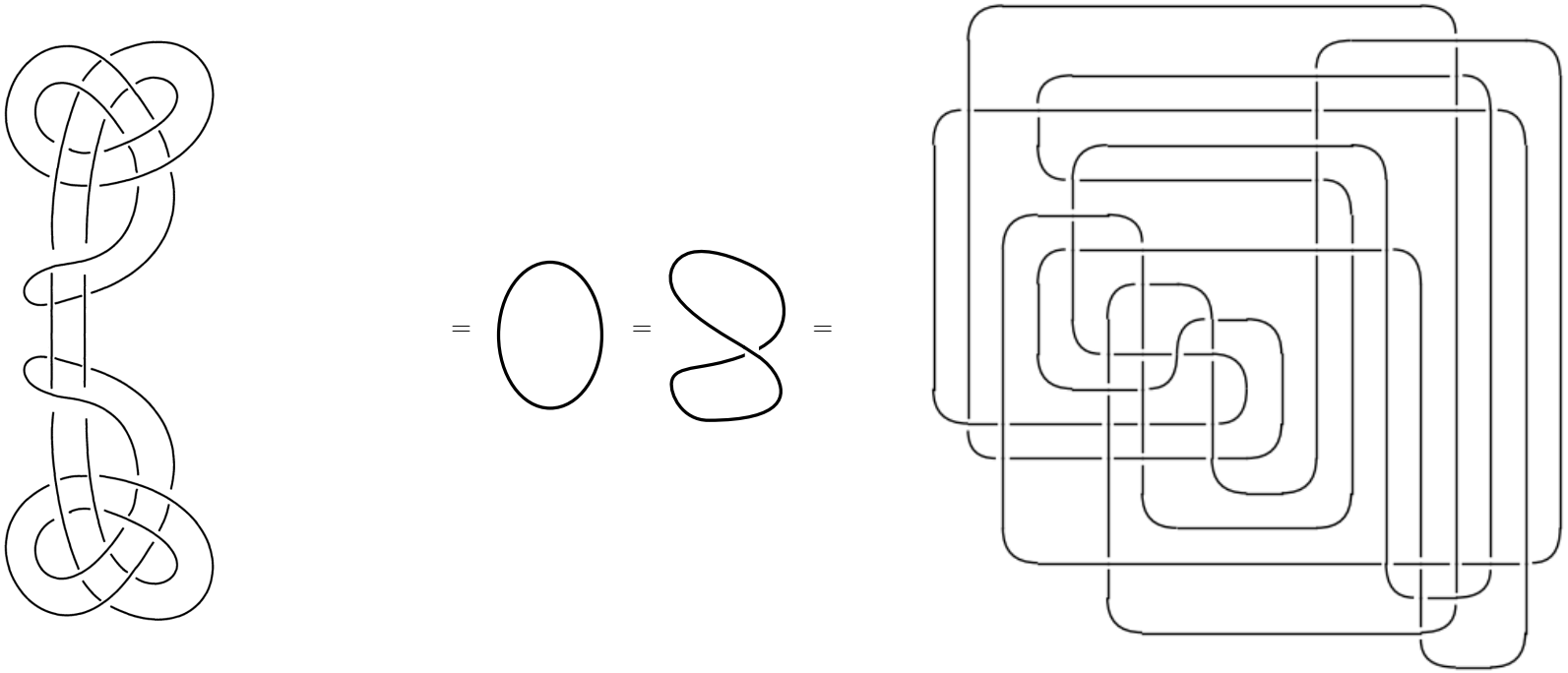




For a diagram with n crossings of the trivial knot:

Theorem (Lackenby 2015)

Their is a sequence of $O(n^{11})$ moves, adding at most $O(n^2)$ crossings, that undo the diagram.



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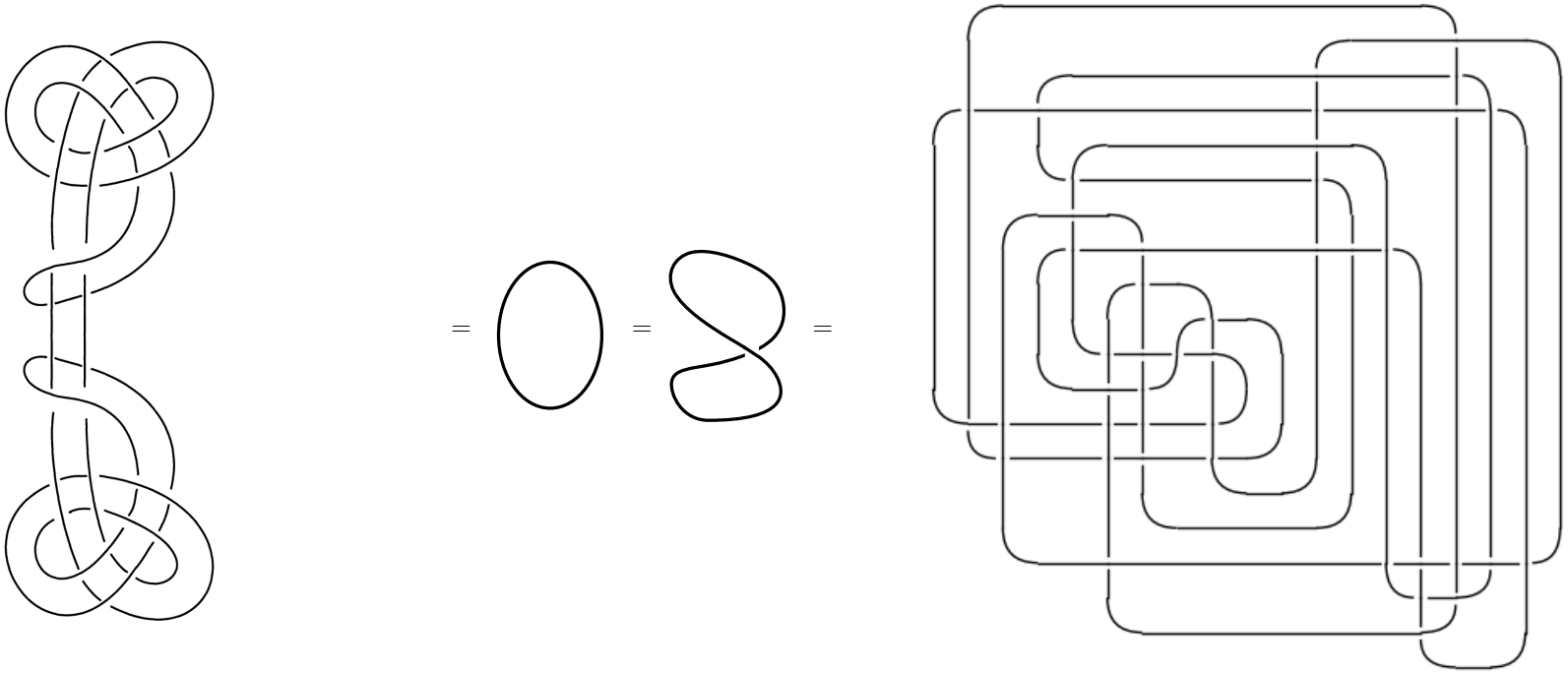

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Theorem (de Mesmay, Rieck, Segwick, Tancer 2019)

It is NP-hard to decide whether a diagram of the unknot can be undone in k moves.

The Jones polynomial

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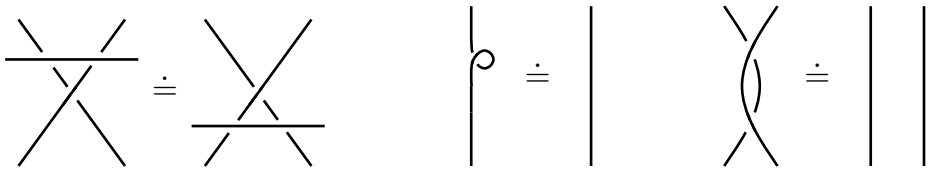

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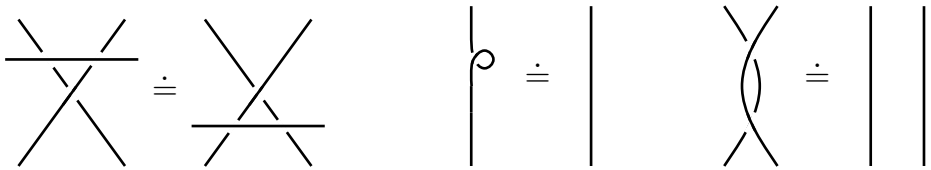

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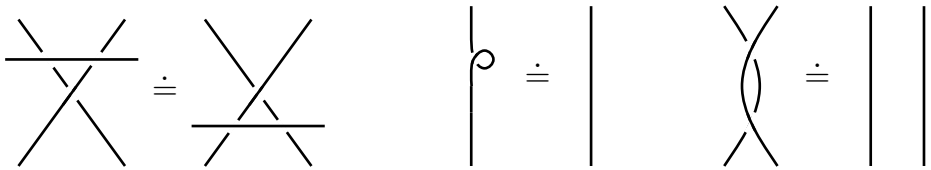

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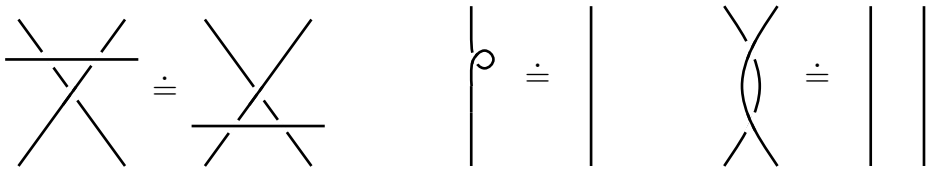

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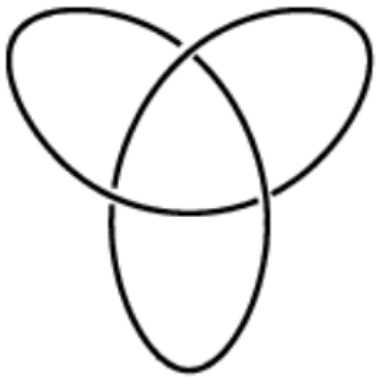
- for the trivial knot: $X(O) = 1$.
- for the trefoil knot: $X(T) = A^{-4} + A^{-12} - A^{-16}$.



Recursive formula using Kauffman bracket

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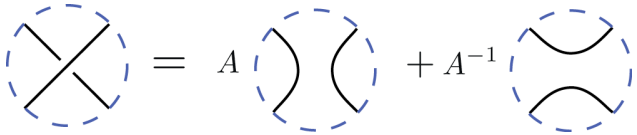
- value for the unknot: $\langle O \rangle = 1$
- recursive relation: 

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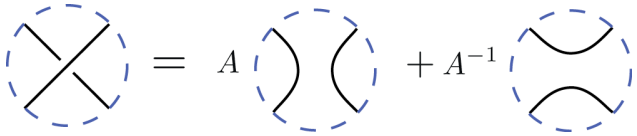

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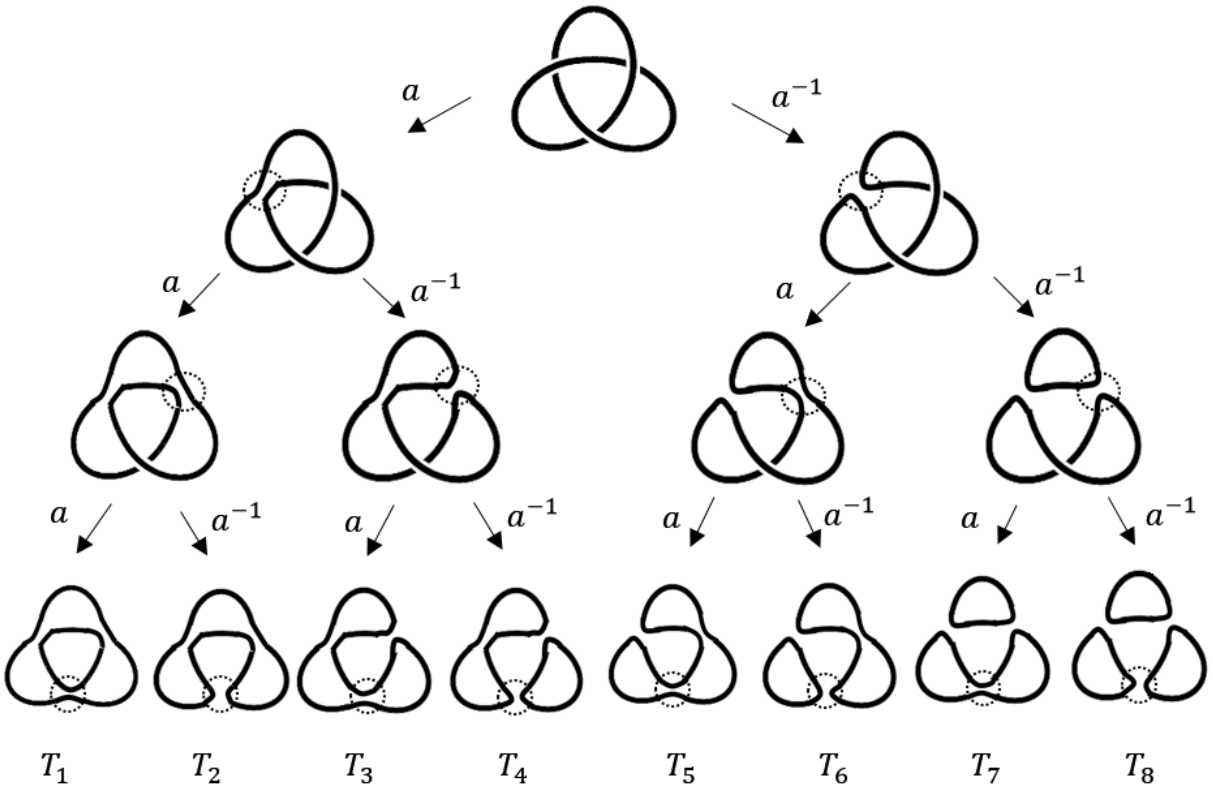

n crossings $\Rightarrow 2^n$ smoothings of the knot.

$2^{O(n)}$ time algorithm

picture from [Li et al. 2016]

Computational complexity

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Other interesting results:

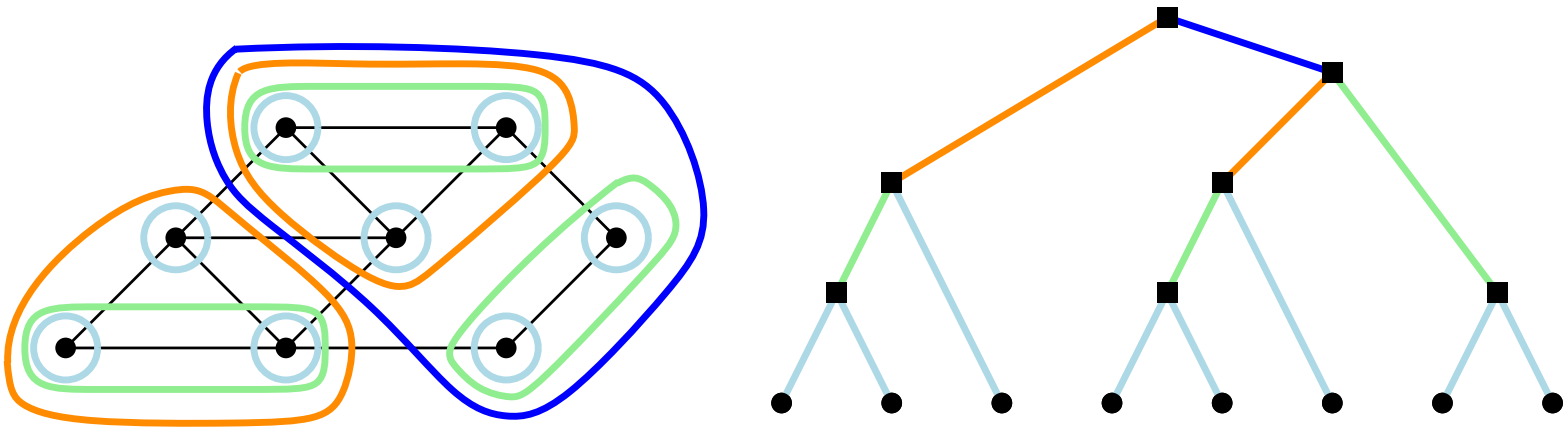
Theorem (Freedman, Kitaev, Larse, Wang (2002), Aharonov, Jones, Landau 2009)

Computing the Jones polynomial is exactly as hard as quantum computing (i.e., BQP-complete).

approximability, evaluation at certain values, etc...

Fast computation: Parameterised complexity

Bond carving decomposition of planar graphs



- System of Jordan curves, cutting the graph transversally,
- The width is the maximal number of intersections between a Jordan curve and the graph edges.

The smallest possible width is the carving-width cw.

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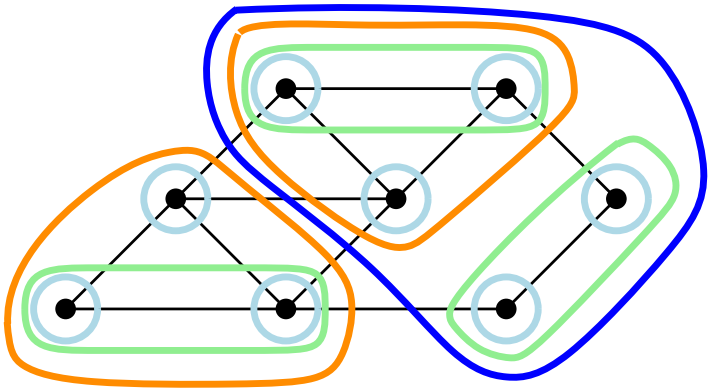

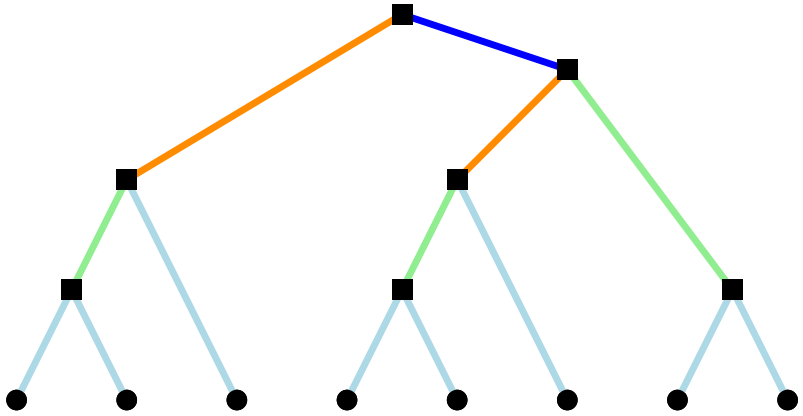

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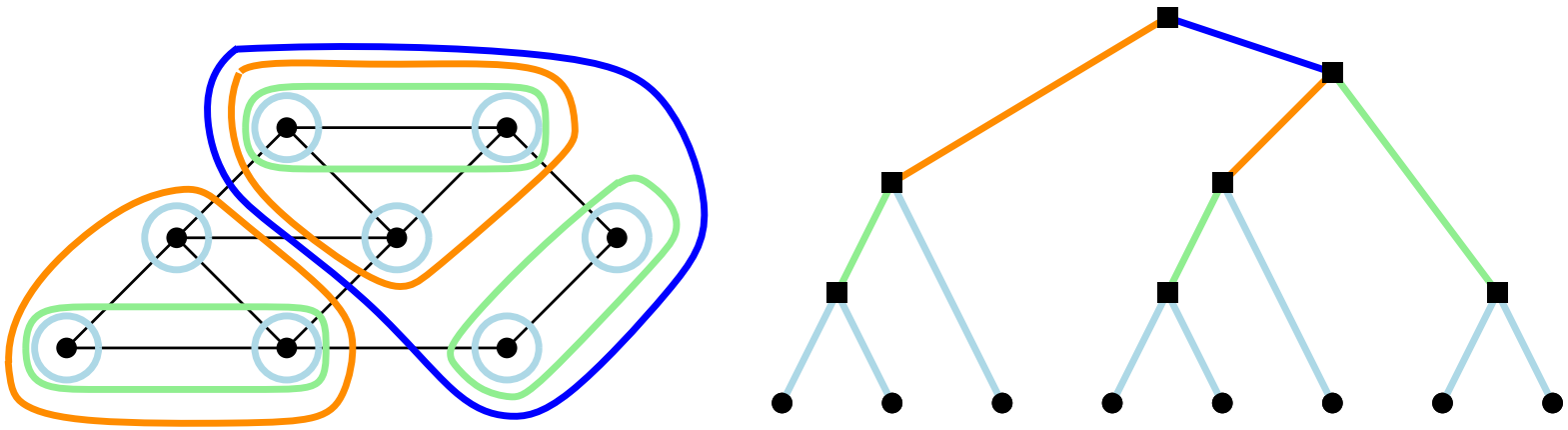

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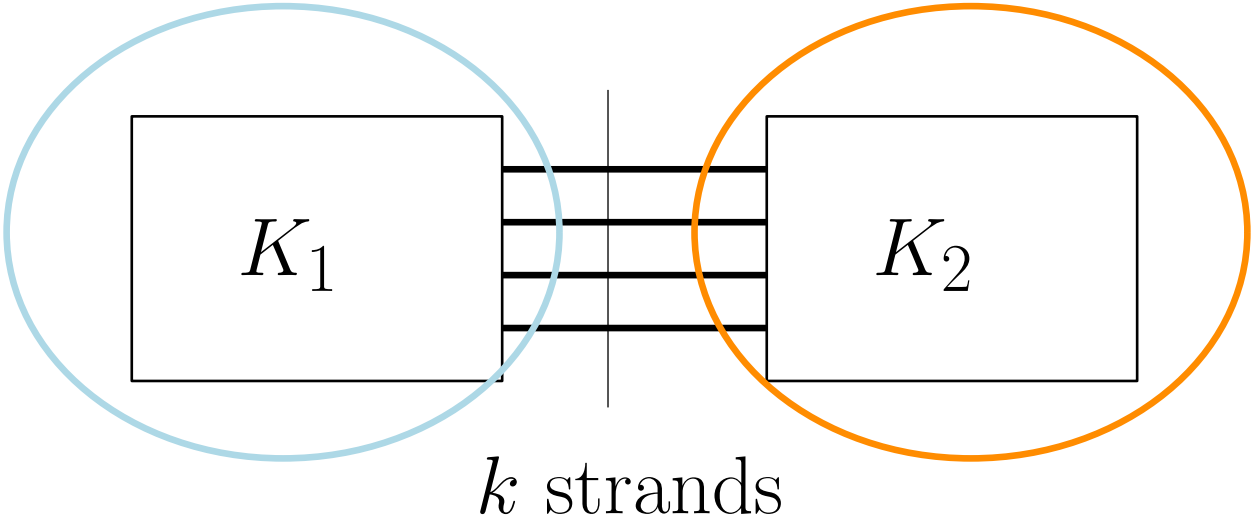
It can be computed in polynomial time on planar graphs.

Parameterized algorithm for Jones, intuition

Theorem (Makowsky, Marino 2003, M. 2019)

There is a $2^{O(cw)}$ time algorithm to compute the Jones polynomial and its generalisations (essentially all "quantum knot invariants").



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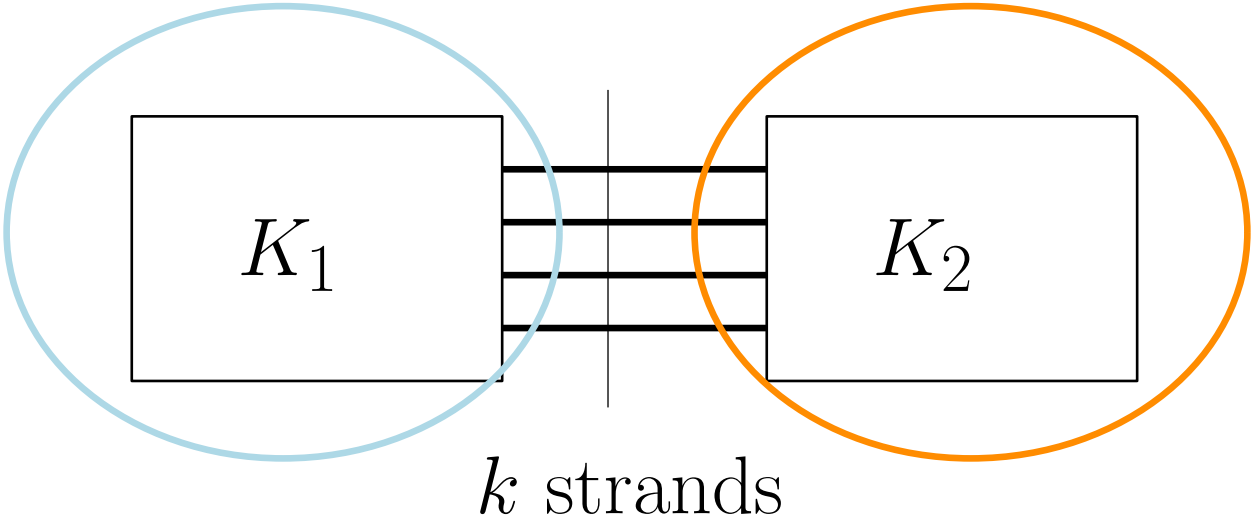
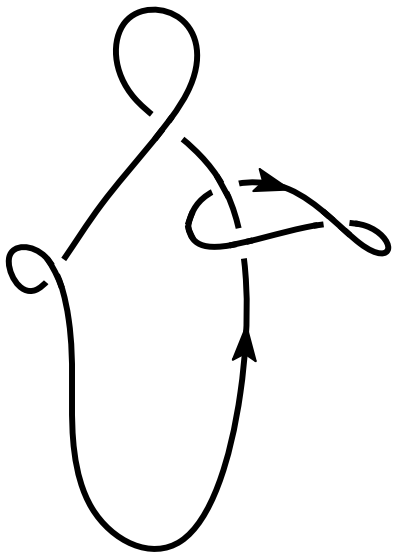


By the planar separation theorem, $\mathrm{cw} \in O(\sqrt{n})$ for an n node planar graph.

This gives a worst case $2^{O(\sqrt{n})}$ time algorithm for the Jones polynomial, and usually much better.

Parameterized complexity for quantum invariants

Penrose functor: Diagram $\to$ invariant



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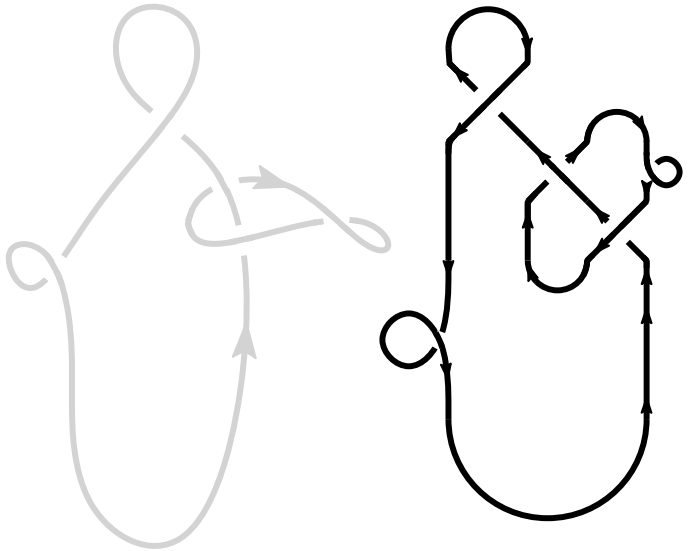

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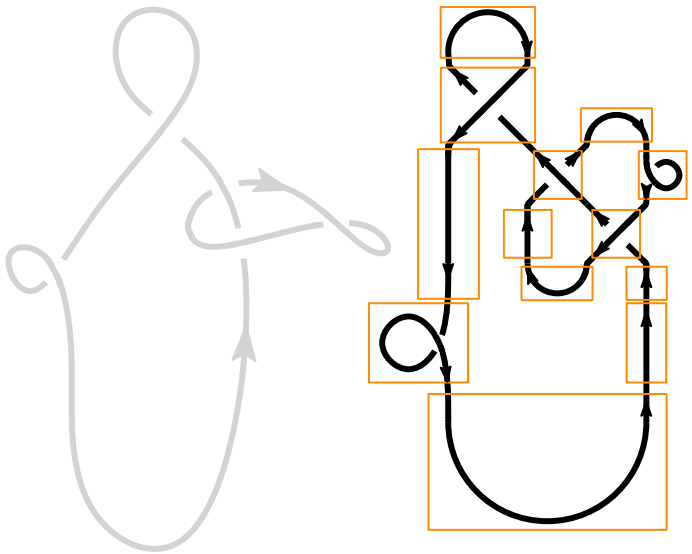

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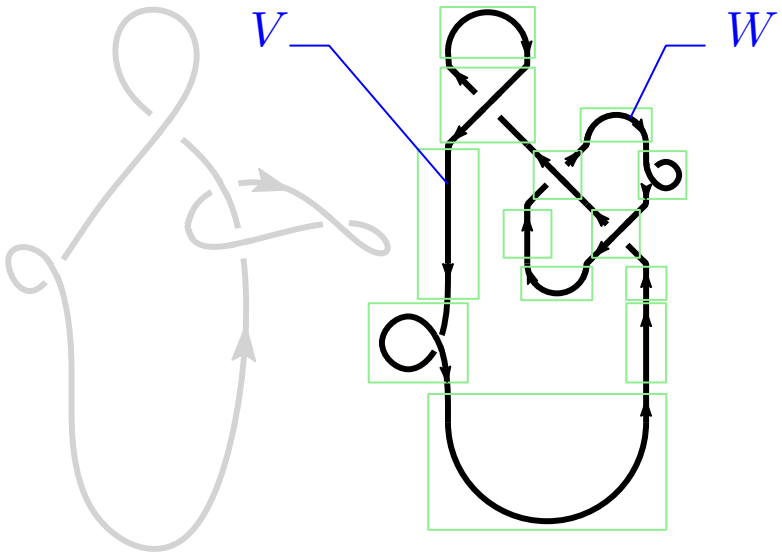

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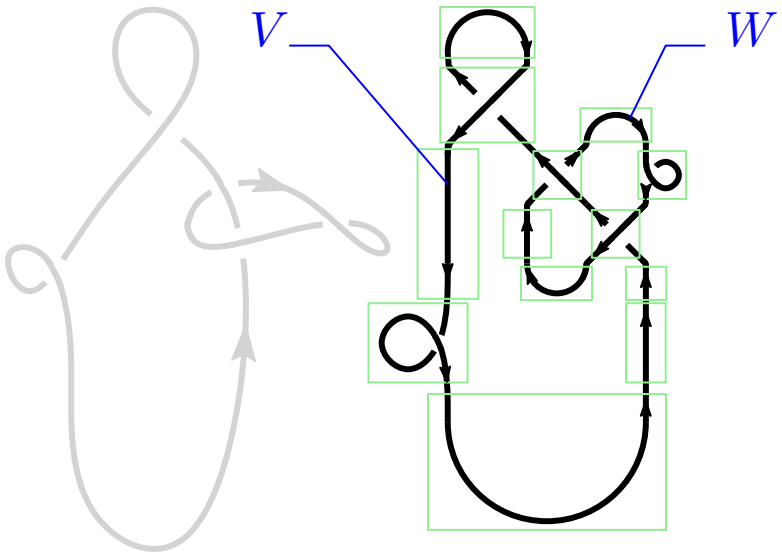

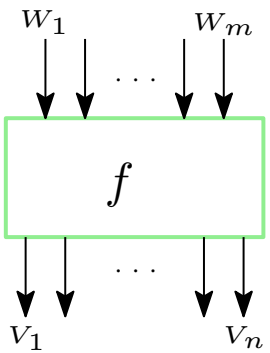

$f\colon V_1 \otimes \ldots \otimes V_n \to W_1 \otimes \ldots \otimes W_m$

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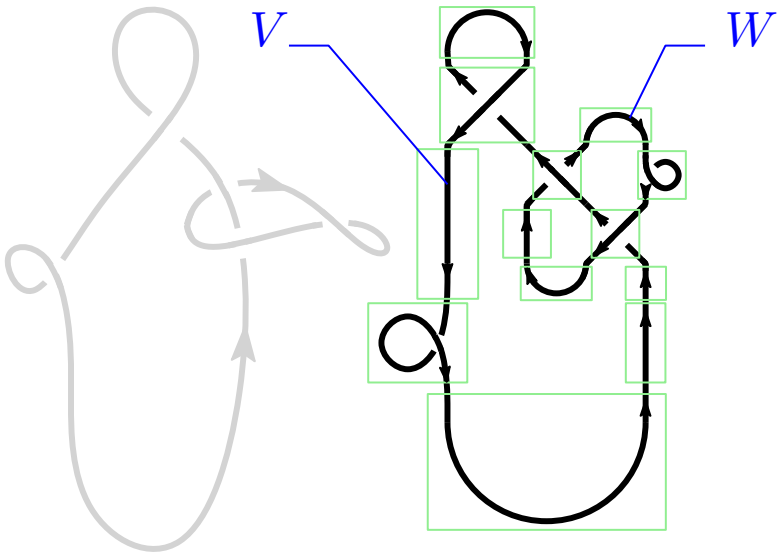

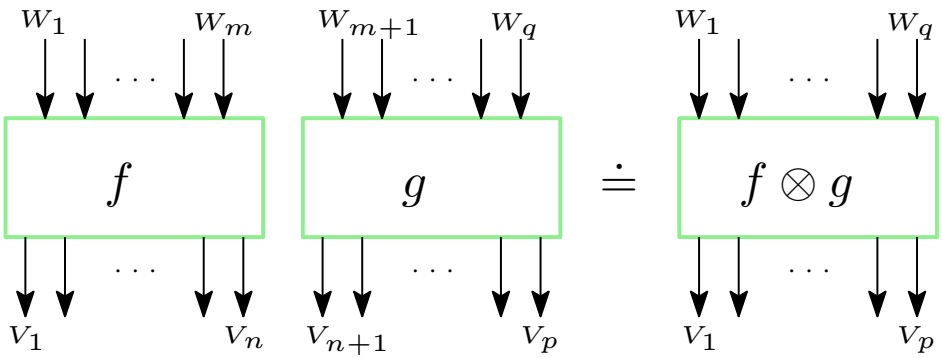

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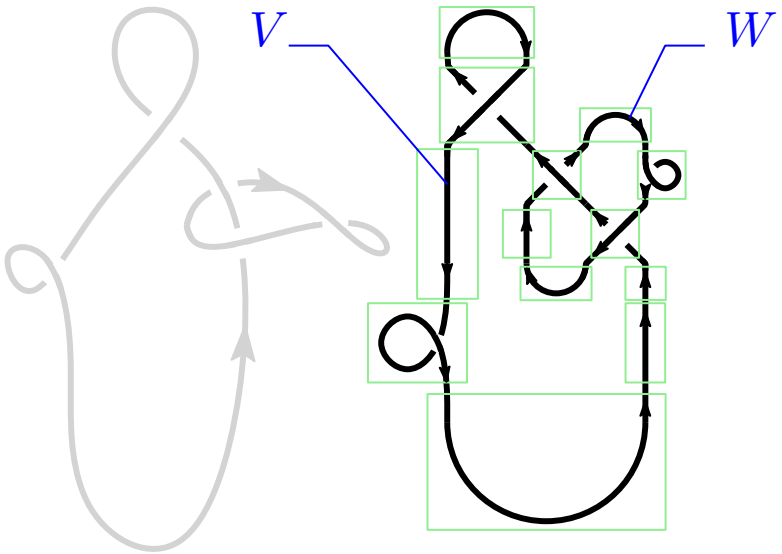

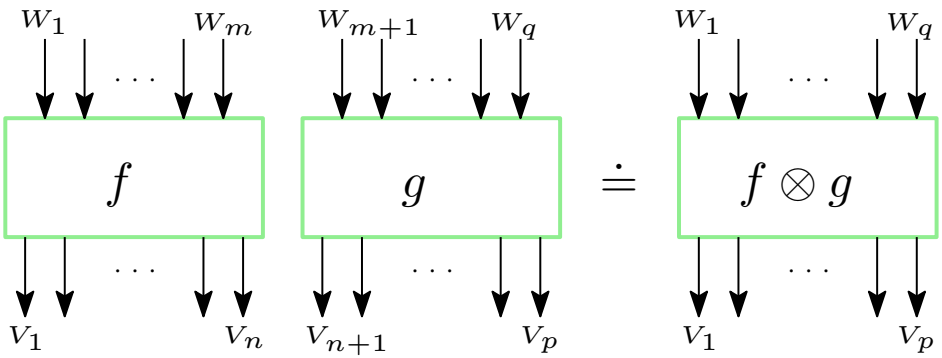

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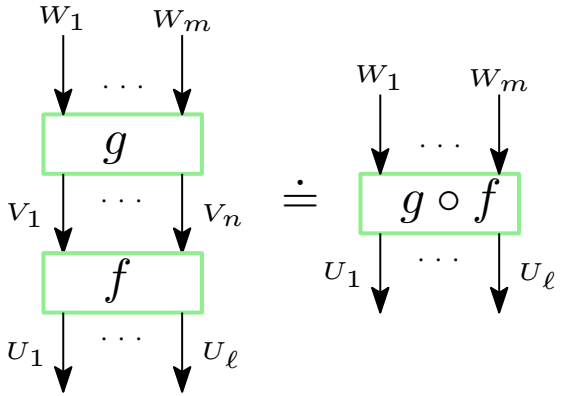

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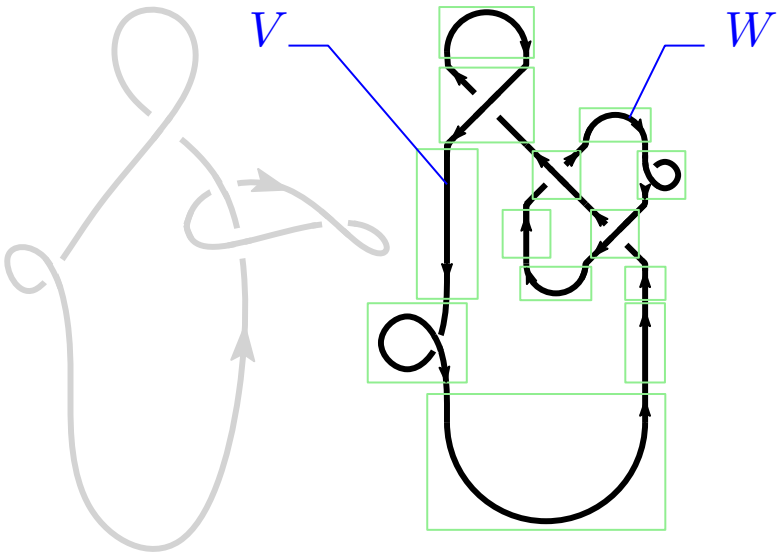

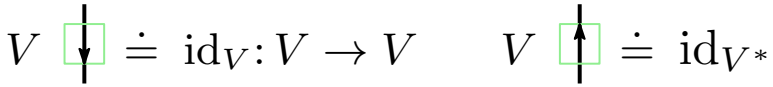

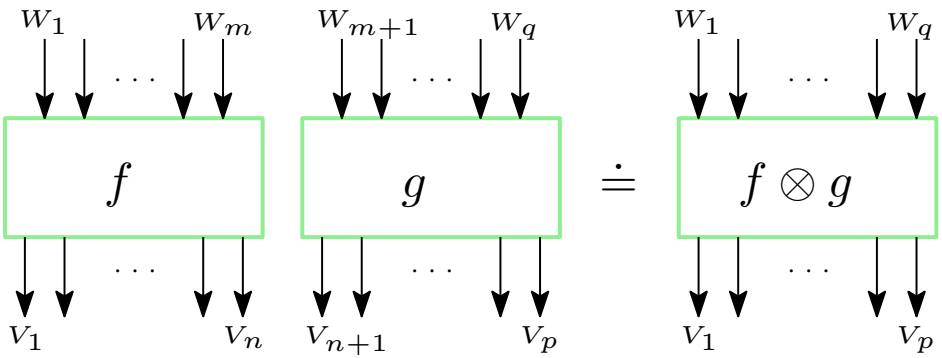

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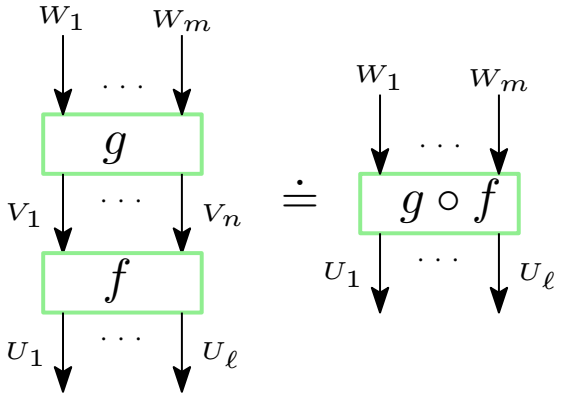

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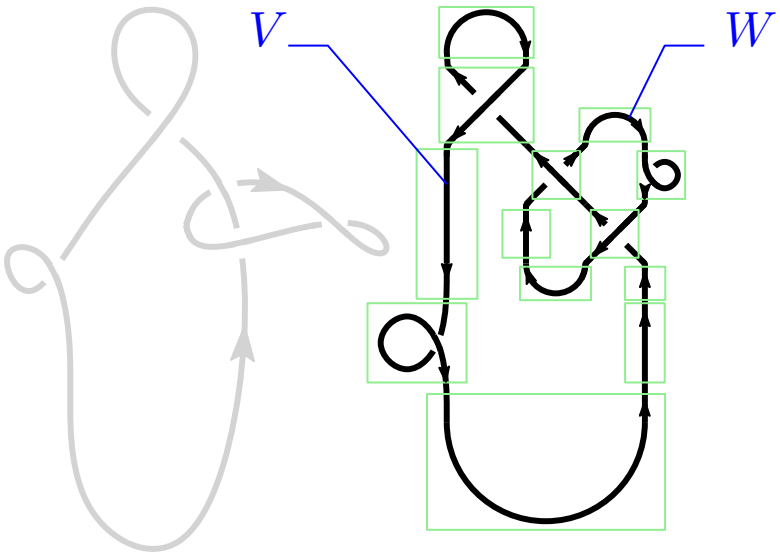

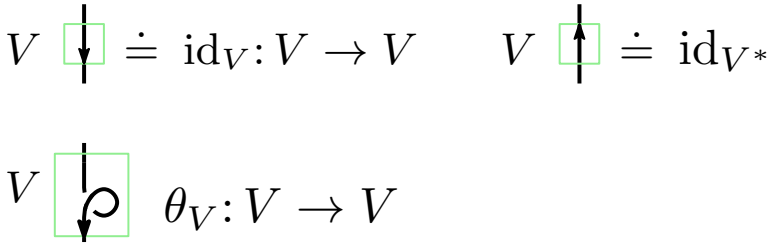

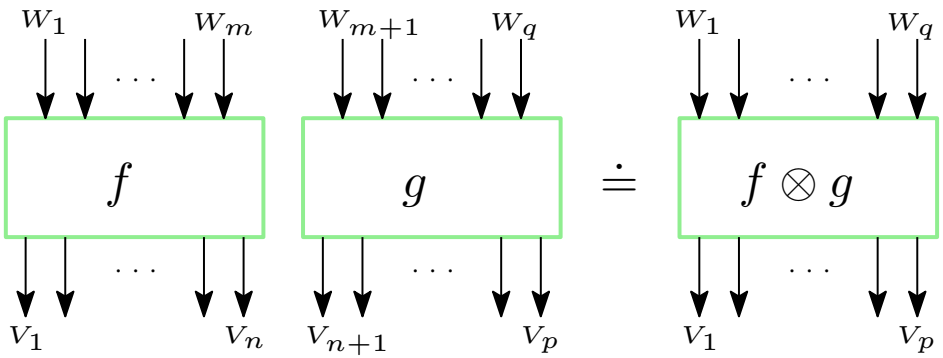

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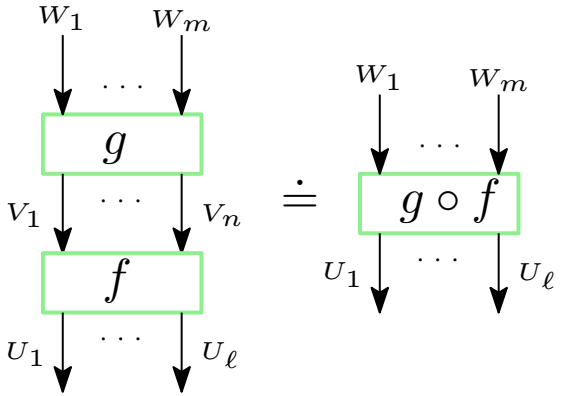

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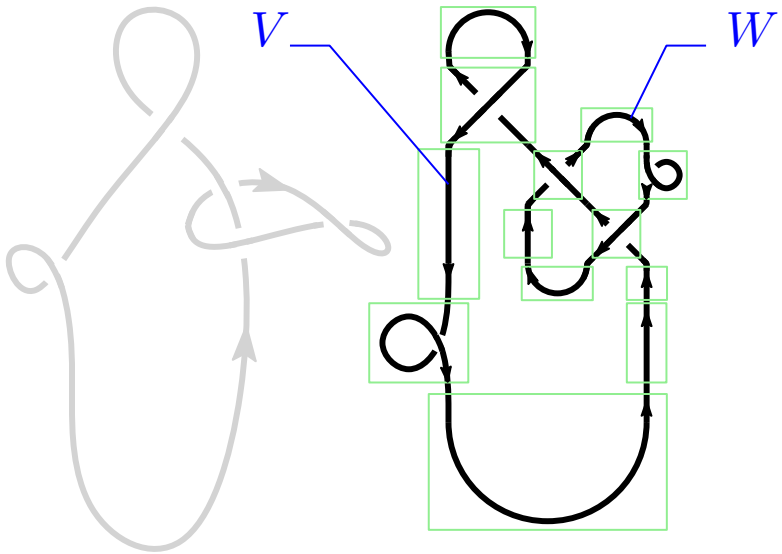
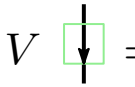
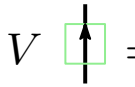
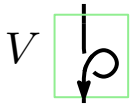




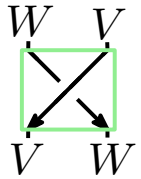
$\doteq \mathrm{id}_V : V \to V$



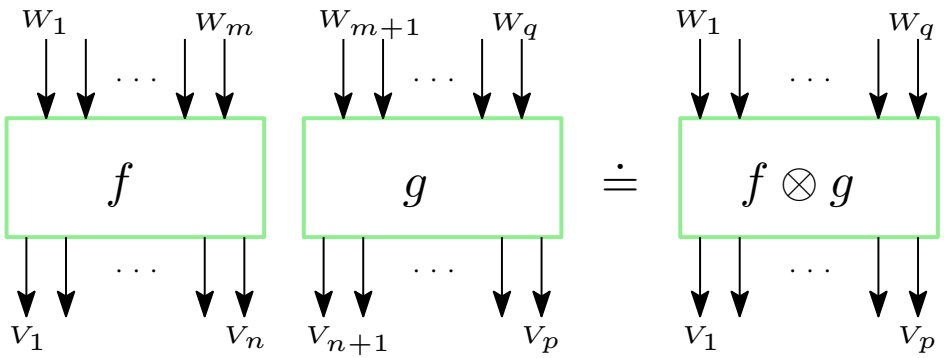
$\doteq \mathrm{id}_{V^*}$



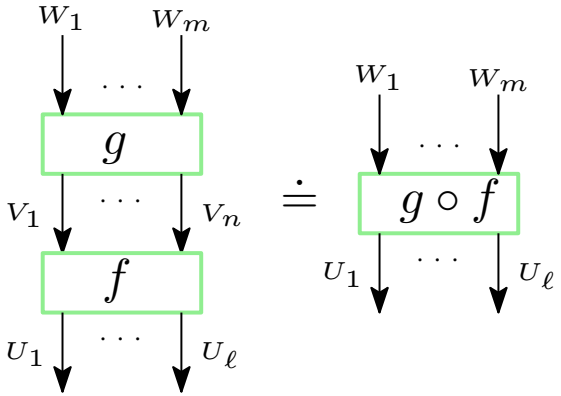
$\theta_V : V \to V$



$c_{V,W} : V \otimes W \to W \otimes V$

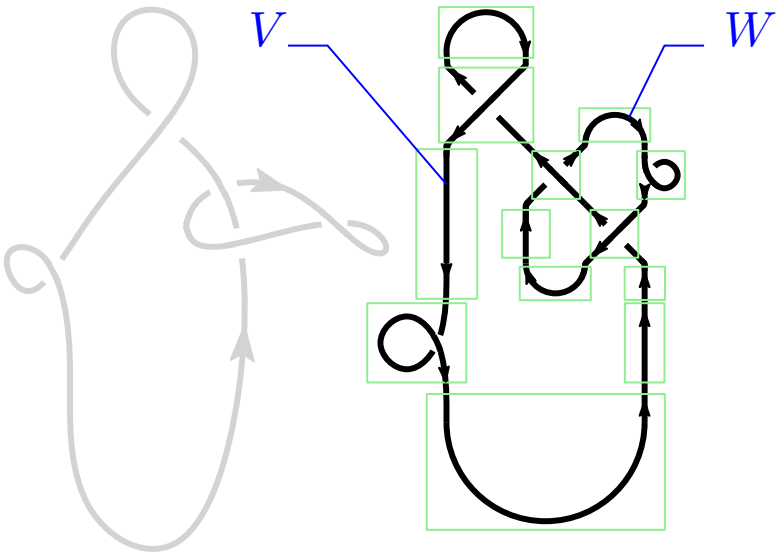


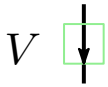
$f \otimes g : V_1 \otimes \ldots \otimes V_p \to W_1 \otimes \ldots W_q$



$g \circ f : U_1 \otimes \ldots \otimes U_\ell$

Penrose functor: Diagram $\to$ invariant





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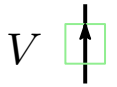

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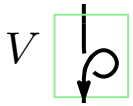

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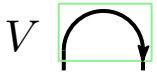

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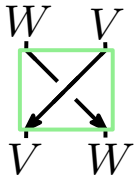

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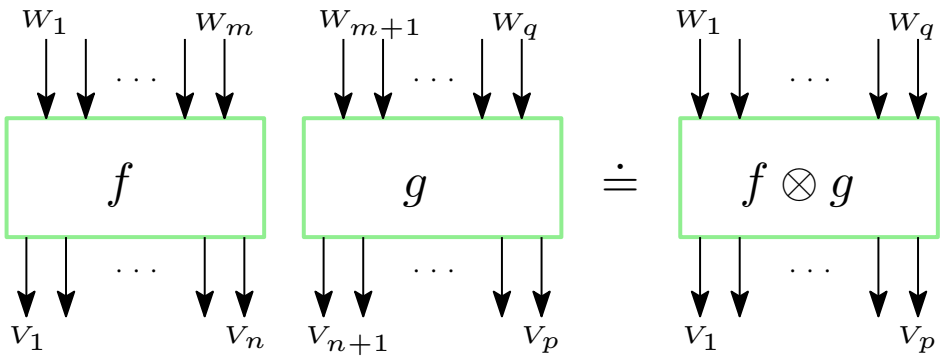

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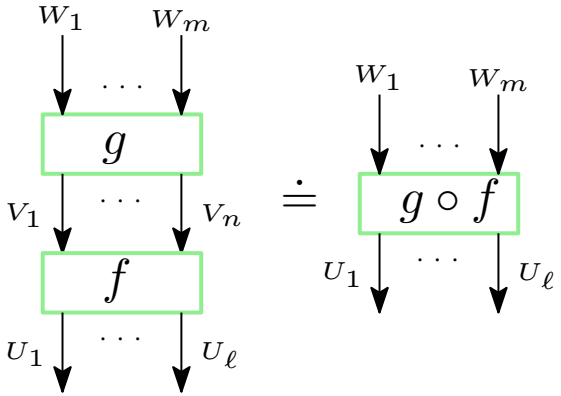

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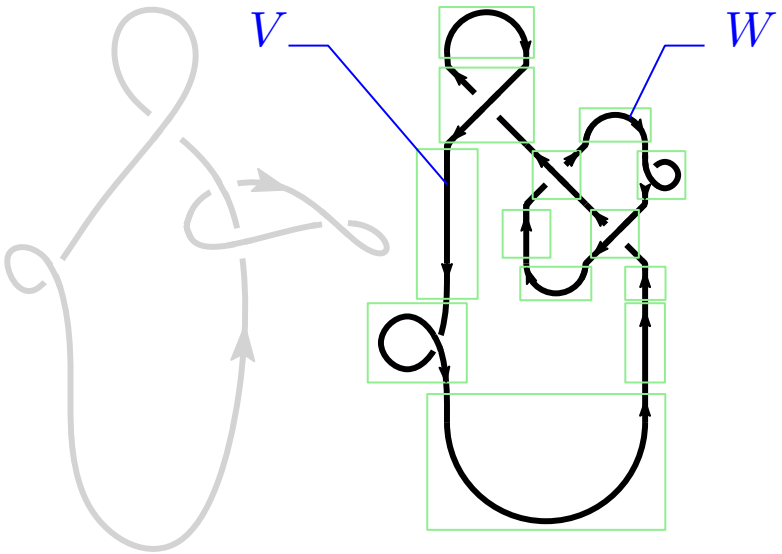

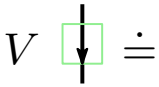

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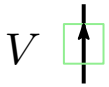

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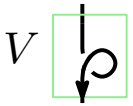

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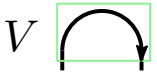

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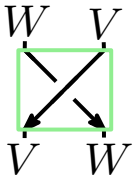

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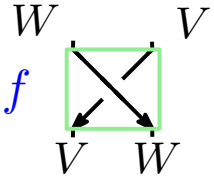
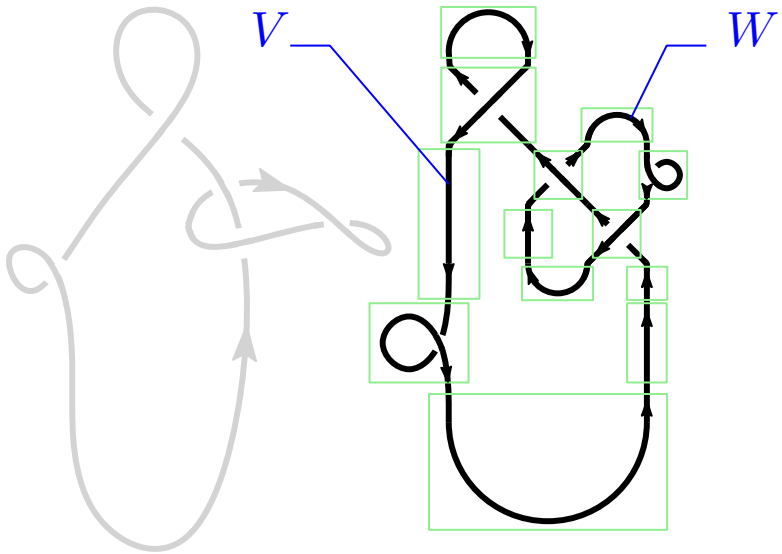
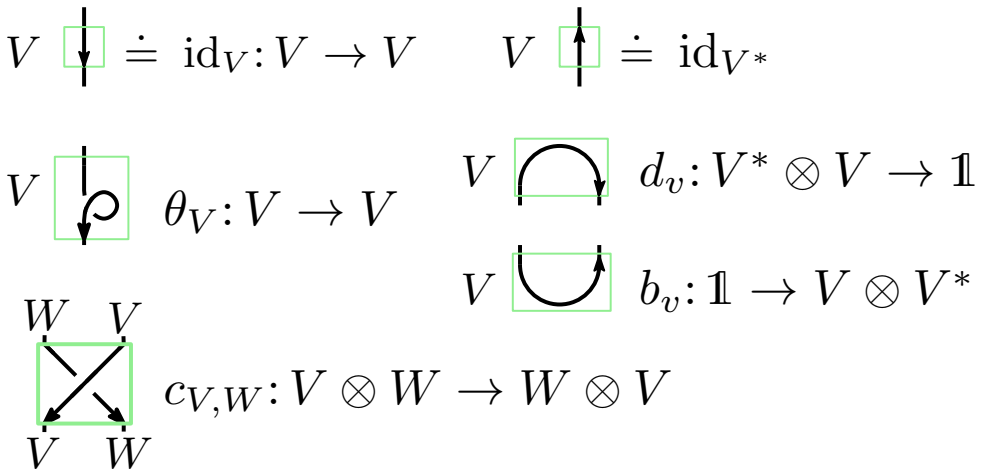
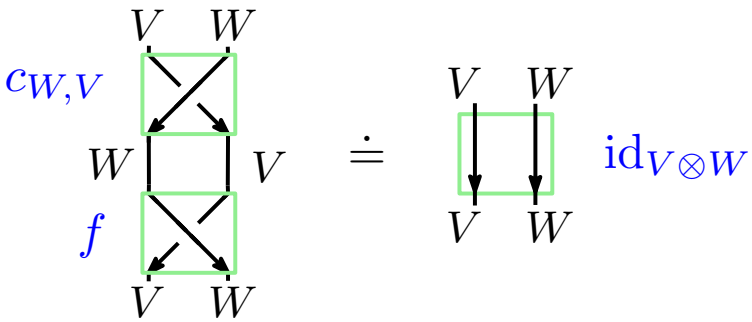
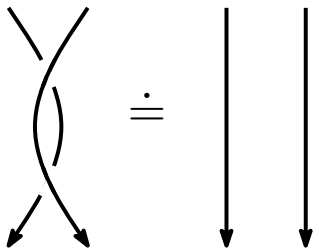


Penrose functor: Diagram $\to$ invariant









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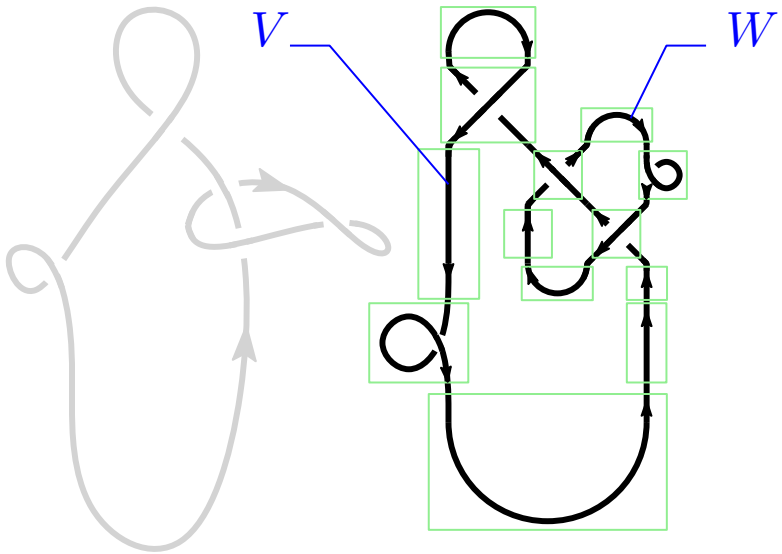

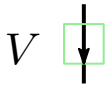

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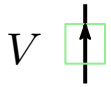

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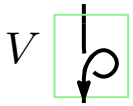

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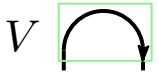

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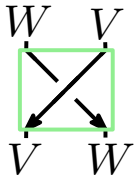

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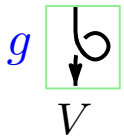

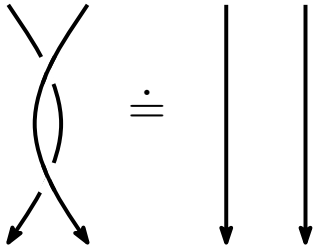

$\doteq$

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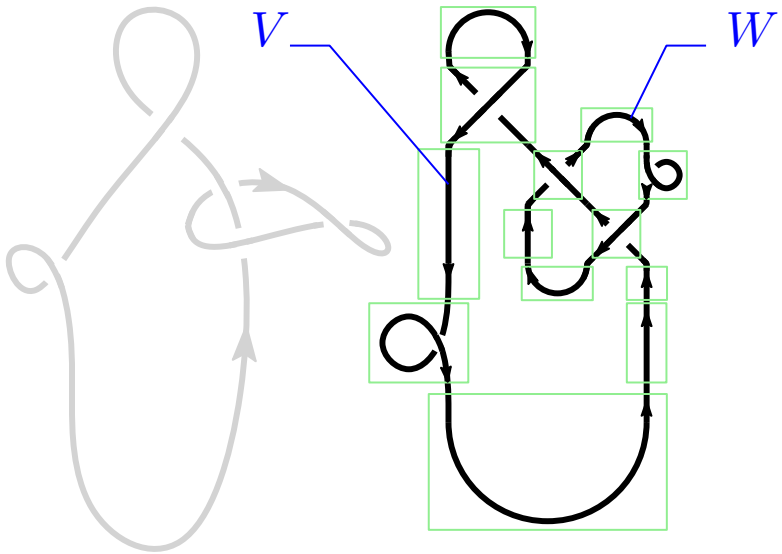

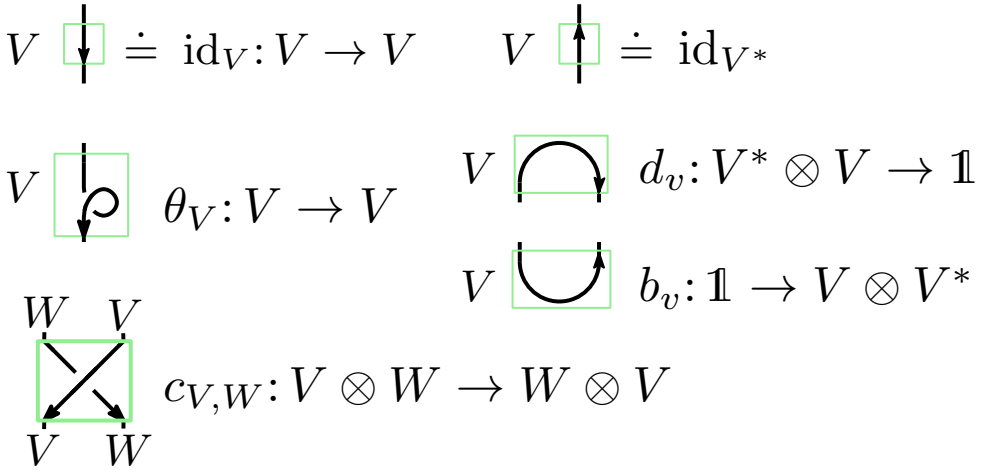

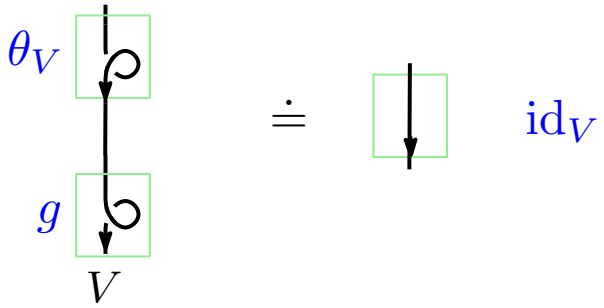

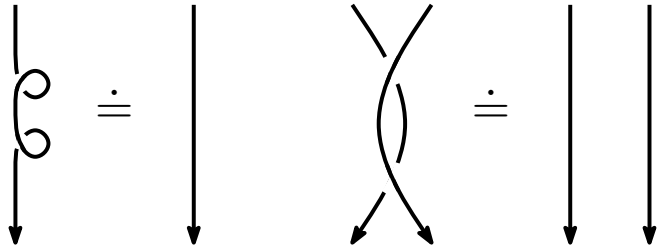

Fixed parameter tractable algorithm

Hypothesis and degrees of freedom

`Input:` A knot diagram, coloured by objects of a ribbon category, with bound carving decomposition.

- Fix a ribbon cat $\mathcal{C}$: Restrict to free R-modules ; morphisms are matrices.

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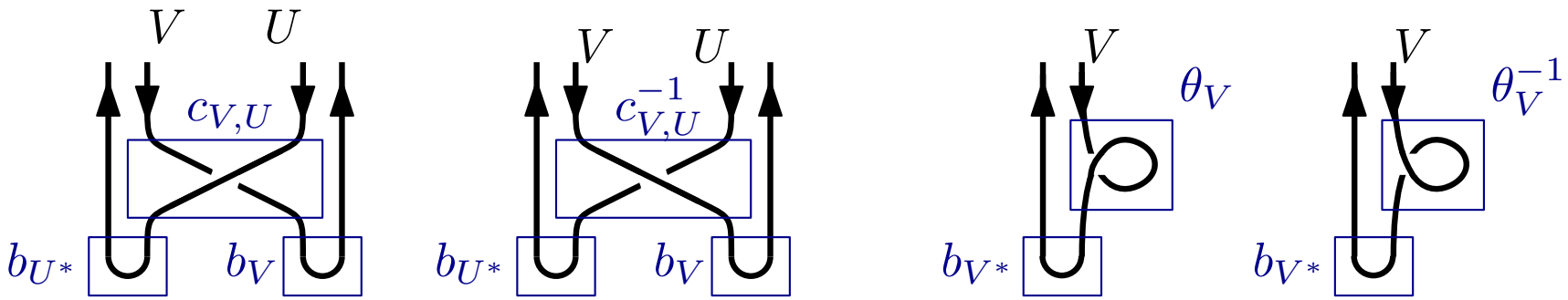
• We are allowed ambient isotopies of the knot, as it does not change the invariant.

Leaves of the carving decomposition

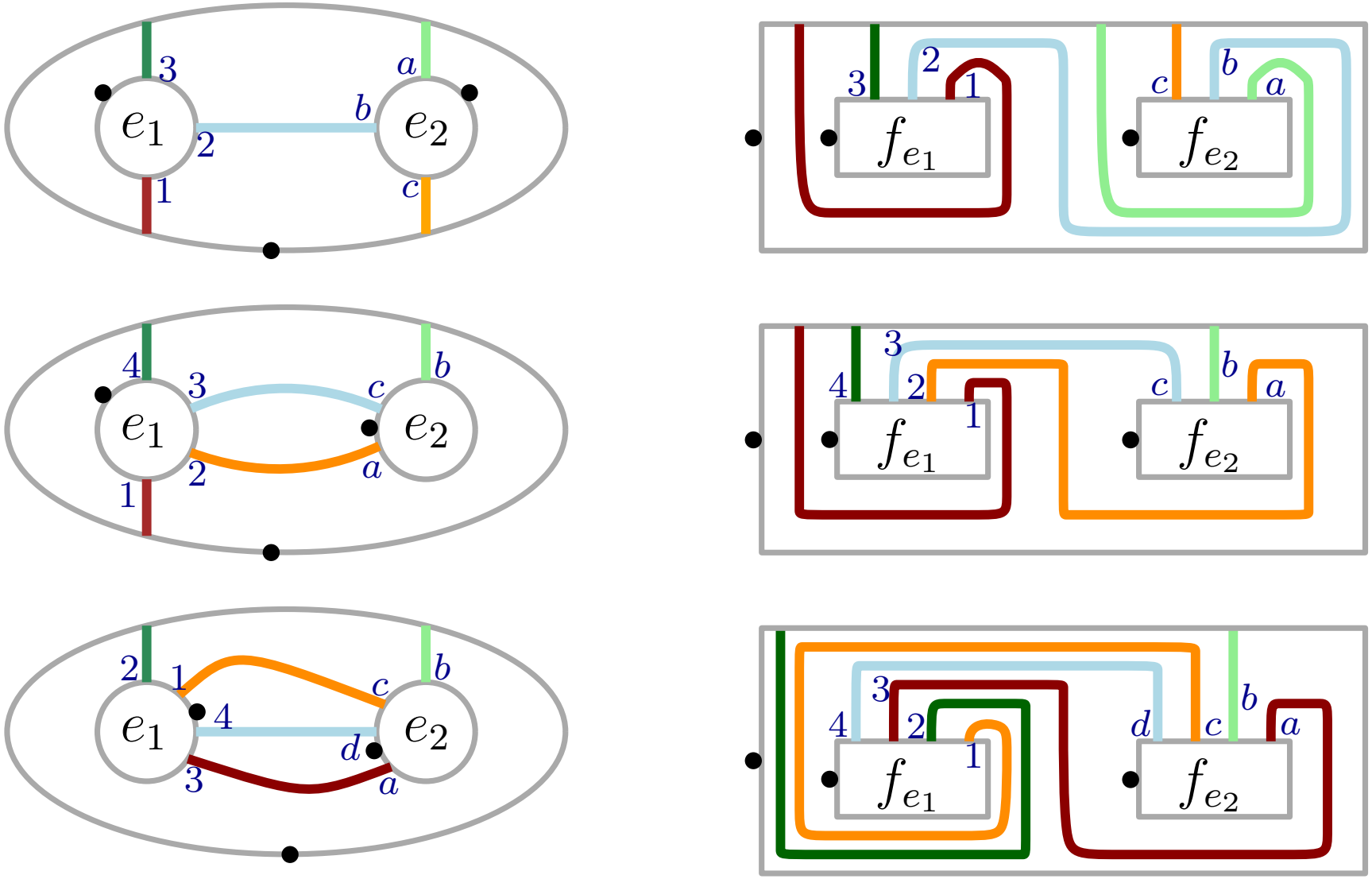
Isotope the link to get morphisms $\mathbb{1} \to W$:



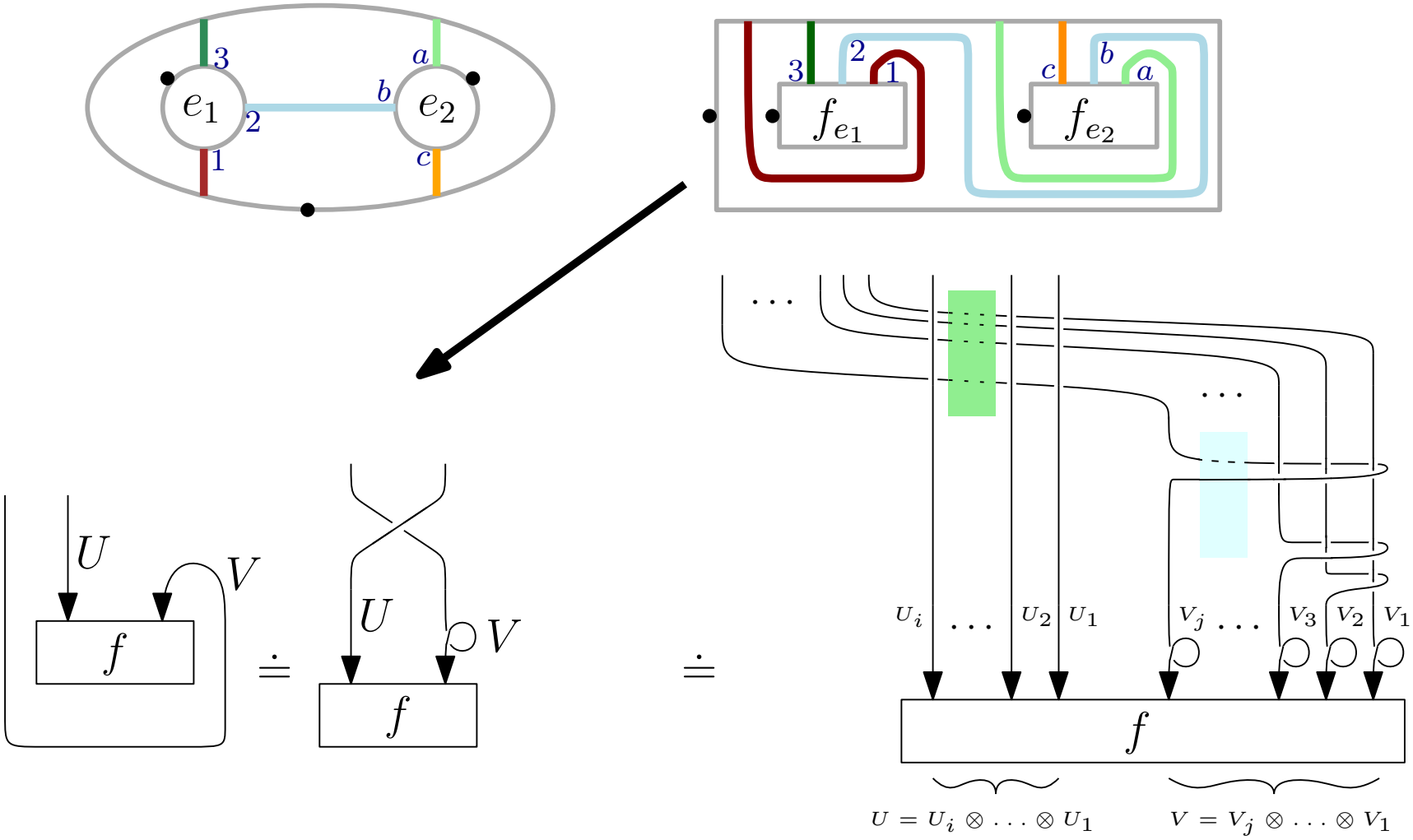
- Four possibilities: two crossings, two twists.
- Constant number of matrix multiplications.

Merging morphisms at tree nodes I

Isotope the link to get morphisms $\mathbb{1} \to W$:

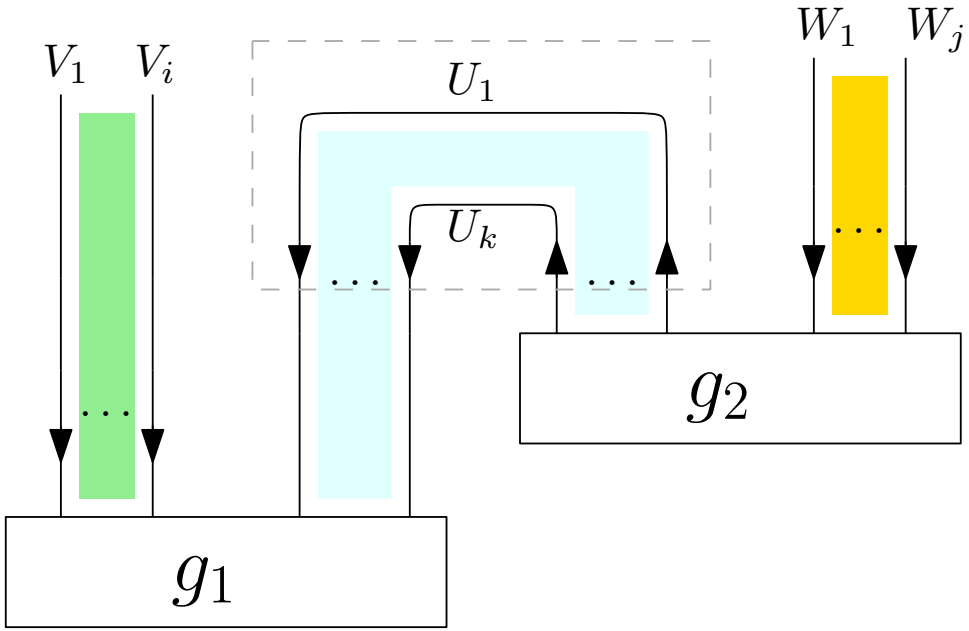


Merging morphisms at tree nodes II



Slide strands under by isotopy $\longrightarrow$ factorise with $O(\mathrm{cw}^2)$ additional matrices.

Merging morphisms at tree nodes III



- $i + j \leq \mathrm{cw}$,
- can assume $k \leq \mathrm{cw}/2$.

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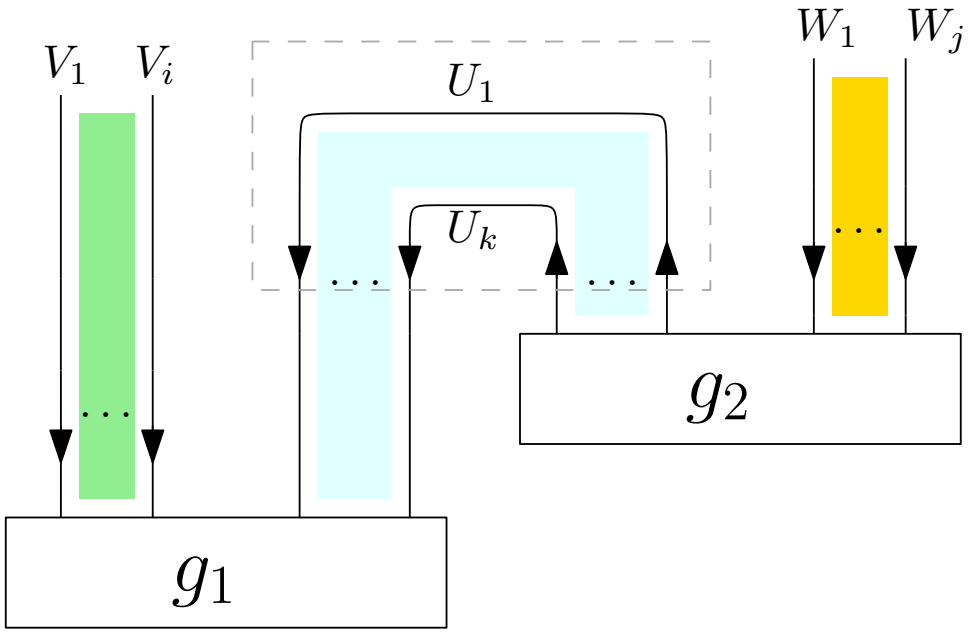

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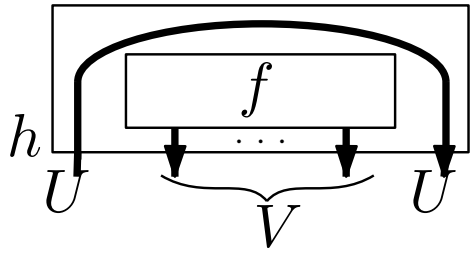
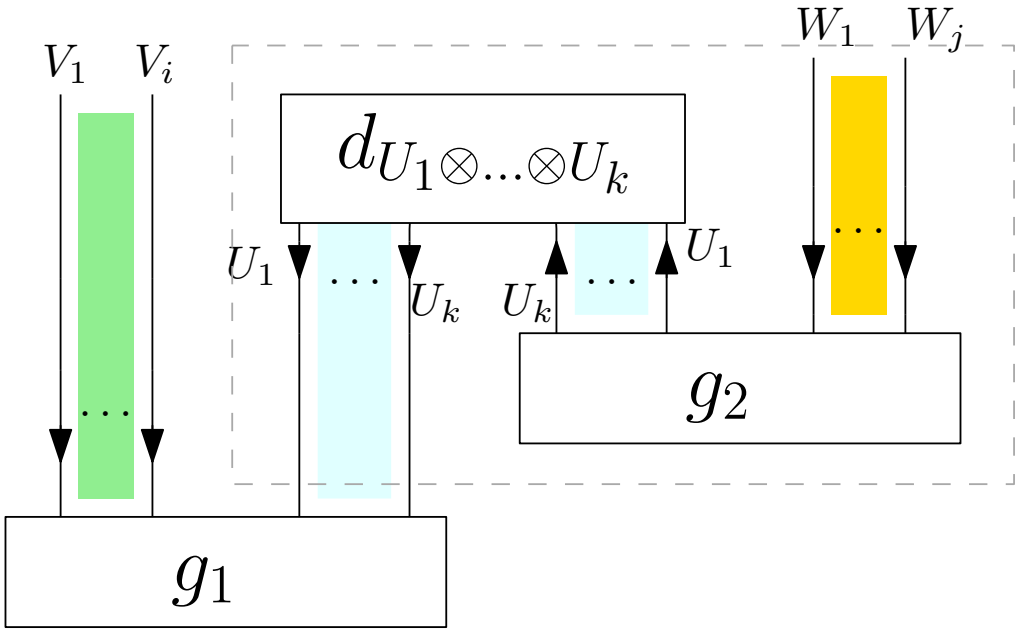
Operation (k times):

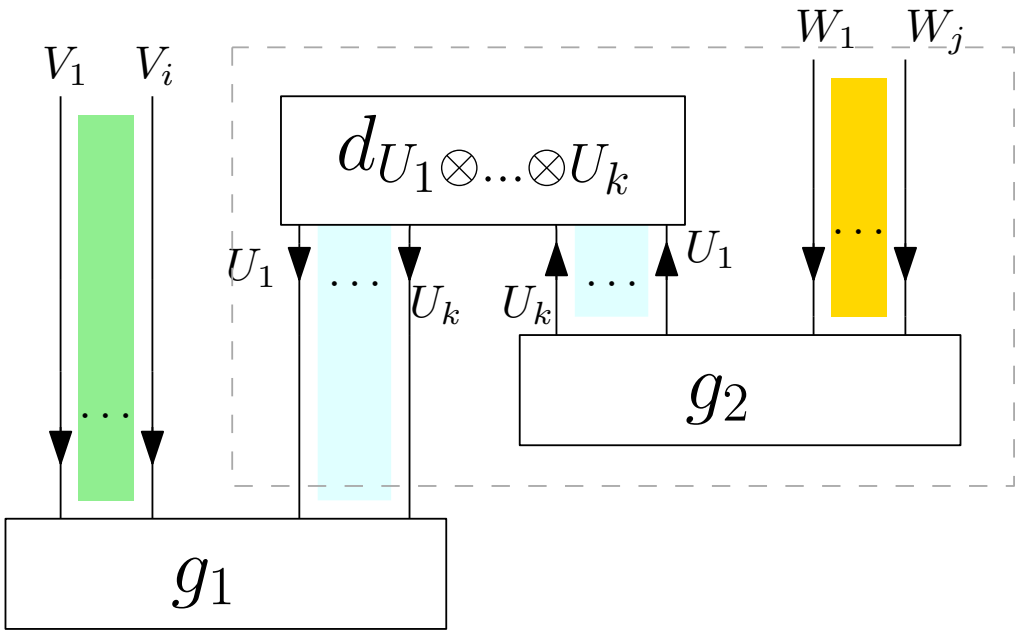


Next $\rightarrow$ factorise the "U_i-bridge".

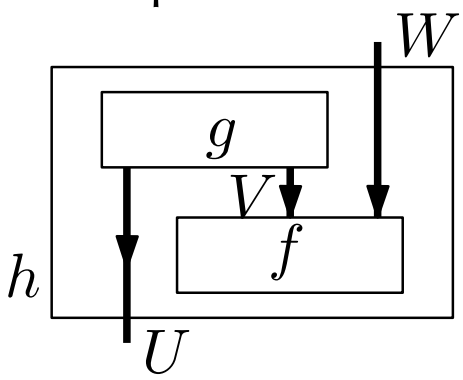
Merging morphisms at tree nodes III



Merging morphisms at tree nodes III

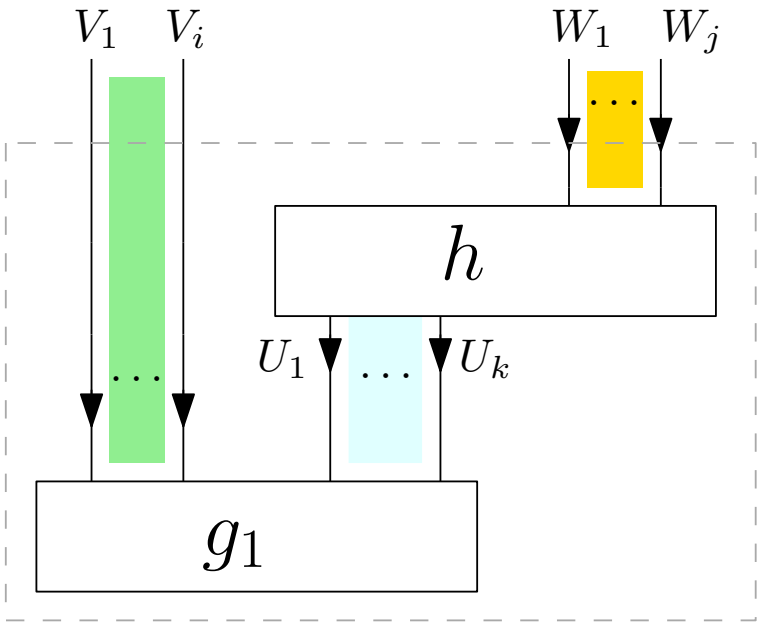


Operation:



Next $\rightarrow$ factorise $d_{U_1 \otimes \ldots \otimes U_k}$ and g_2.

Merging morphisms at tree nodes III



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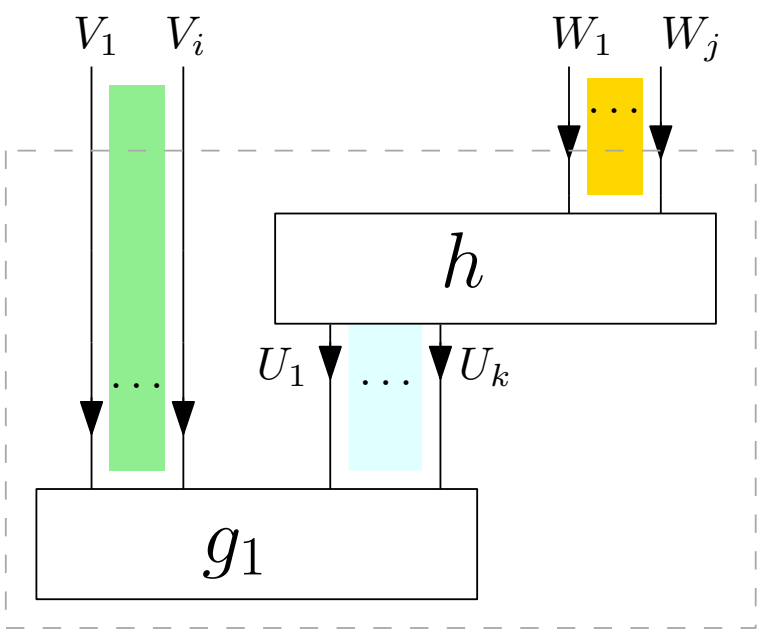

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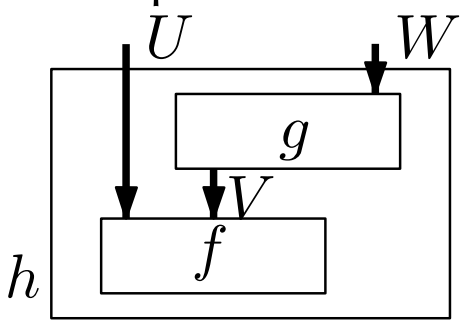
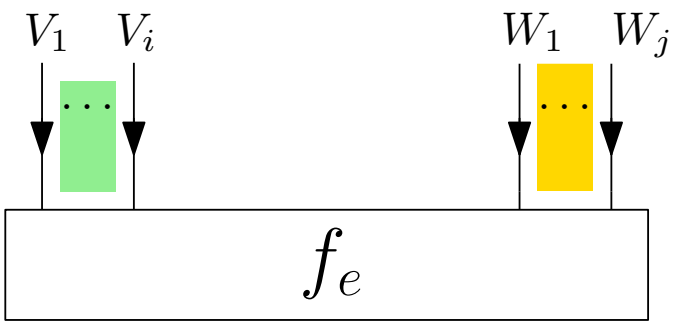
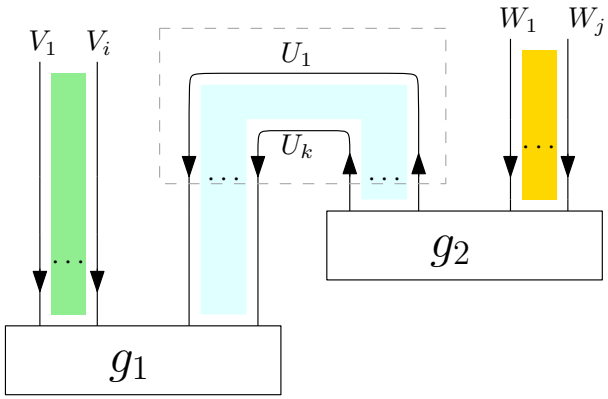


Next $\rightarrow$ factorise h and g_1.

Merging morphisms at tree nodes III



Complexity



- $i + j \leq \mathrm{cw}$,
- can assume $k \leq \mathrm{cw}/2$.

- All op. are sparse matrix multiplications, of type: $\mathrm{id} \otimes M \otimes \mathrm{id}$.

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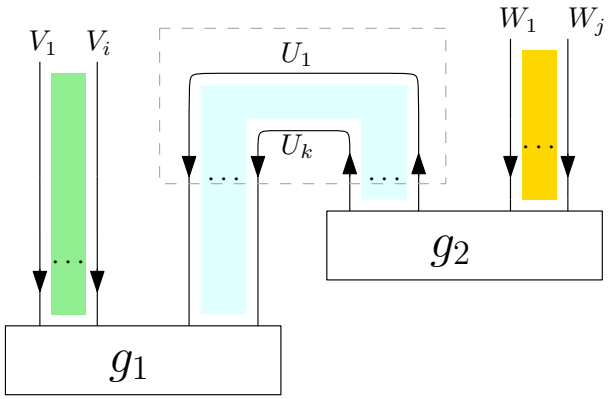

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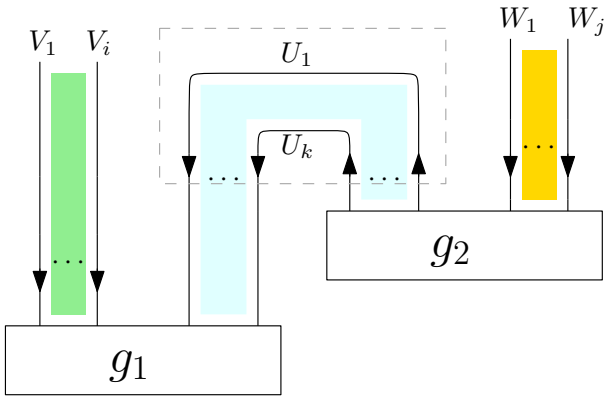

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- Can control the arithmetic complexity of operations in the ring R (e.g., $R = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$).

Complexity

Theorem

Fix a strict ribbon category $\mathcal{C}$ of $\mathbb{Z}[X, X^{-1}]$-modules, and free modules $V_1, \ldots, V_m \in \mathcal{C}$ of dimension bounded by N. The problem:

> Quantum invariant at $\mathcal{C}, V_1, \ldots, V_m$:
> **Input**: *m-components link L, presented by a diagram $D(L)$,*
> **Output**: *quantum invariant $J_L^{\mathcal{C}}(V_1, \ldots, V_m)$*

can be solved in

- $O(\mathrm{poly}(n) \cdot N^{\frac{3}{2}\,\mathrm{cw}})$ *machine operations, with*
- $O(N^{\mathrm{cw}} + n)$ *memory words,*

where n and cw are respectively the number of crossings and the carving-width of the diagram $D(L)$.

NB: $\mathrm{cw} = O(\sqrt{n}) \Rightarrow$ sub-exponential algo.

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Thank you!