Approximation Algorithms for Rectangle Packing

Arindam Khan
IISc Bangalore

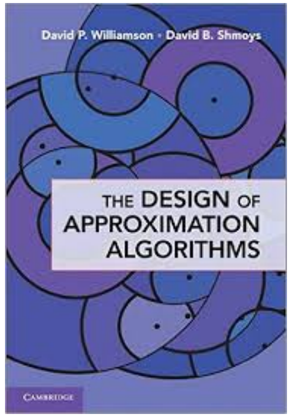
Talk Overview

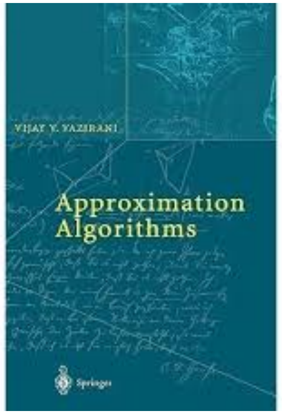
- Approximation Algorithms.
- Bin Packing and Knapsack.
- Rectangle Packing Problems:

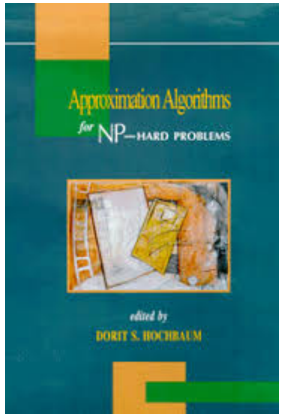
1. 2D Geometric Bin Packing (2BP),
2. 2D Strip Packing (2SP),
3. Dynamic Storage Allocation (DSA),
4. 2D Geometric Knapsack (2GK),
5. Unsplittable Flow on a Path (UFP),
6. Storage Allocation Problem (SAP),
7. Maximum Weight Independent Set of Rectangles (MISR).

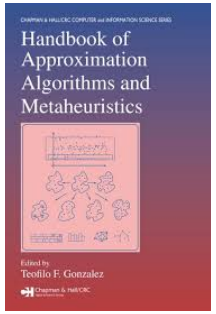
Approximation Algorithms

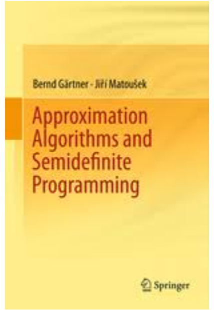
- Approximation algorithms are efficient algorithms that find near-optimal solution.
- For a minimization problem, an algorithm A is α-(absolute) approximation (α>1) if $A(I) \ \leq \ \alpha \, OPT(I)$ for all input instances I .
- For a minimization problem, an algorithm A is α-asymptotic approximation (α>1) if $A(I) \ \leq \ \alpha \, OPT(I) \ + O(1)$ for all input instances I .

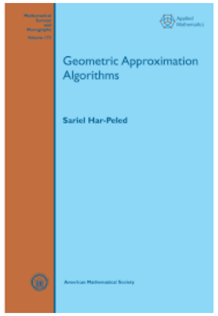


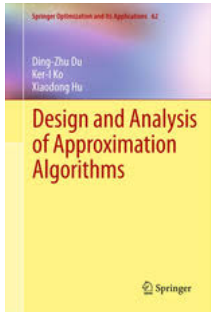










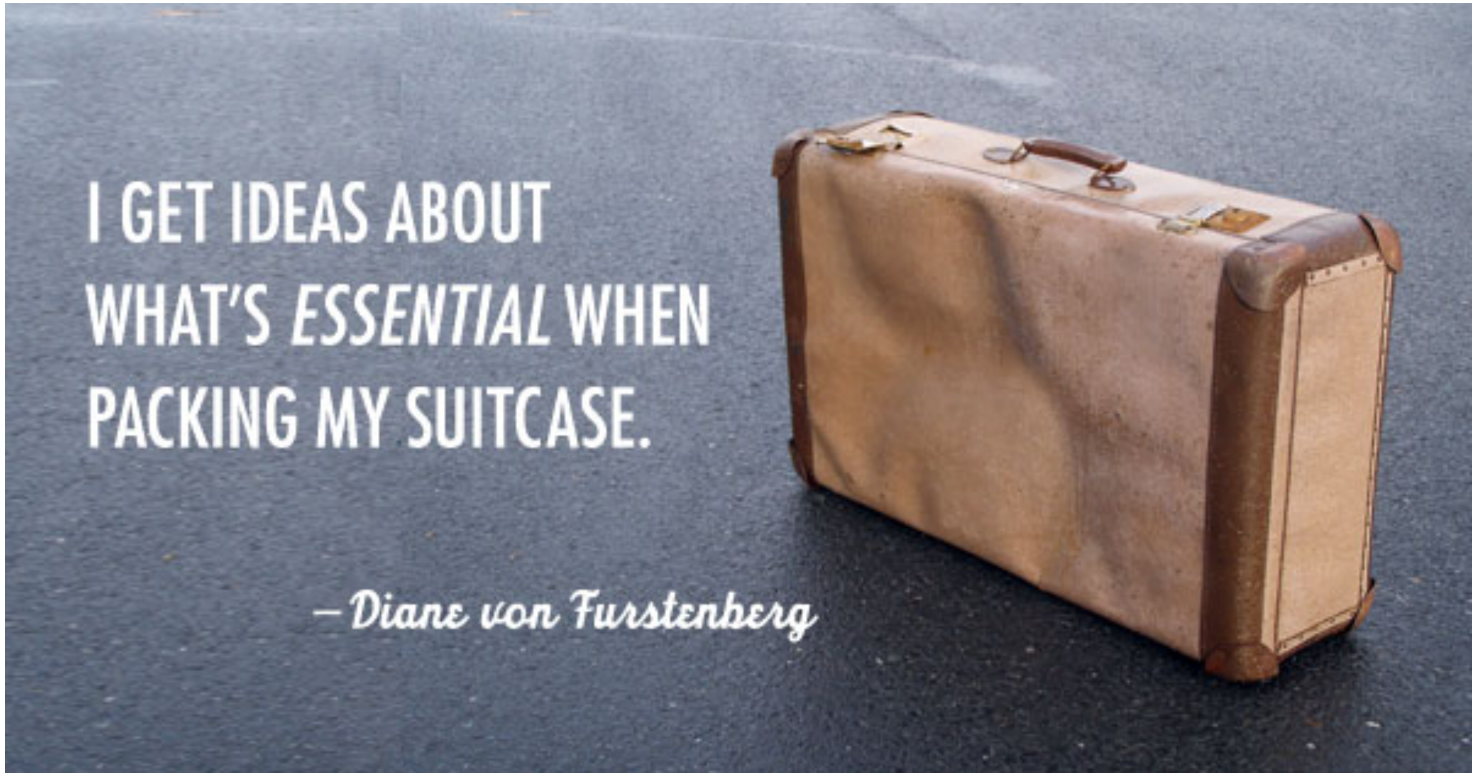


PTAS

PTAS

- Polynomial Time Approximation Schemes (PTAS):
  If for every $\varepsilon > 0$, there exists a poly-time ($O(n^{f(\varepsilon)})$-time) algorithm A_ε such that $A_\varepsilon(I) \le (1+\varepsilon)\, OPT(I)$.
- Efficient PTAS (EPTAS): if running time is $O(f(\varepsilon).n^c)$.
- Fully PTAS (FPTAS): if running time is $O((n/\varepsilon)^c)$.
- Asymptotic PTAS (APTAS): $A_\varepsilon(I) \le (1+\varepsilon)\, OPT(I) + O(1)$.
- QuasiPTAS (QPTAS): $(1+\varepsilon)$-approximation in $n^{(\log n)^{O(1)}}$-time.
- PseudoPTAS (PPTAS): $(1+\varepsilon)$-approximation in $n^{O(1)}$-time, where n is the number of items and the numeric data is polynomially bounded in n.

Packing Problems



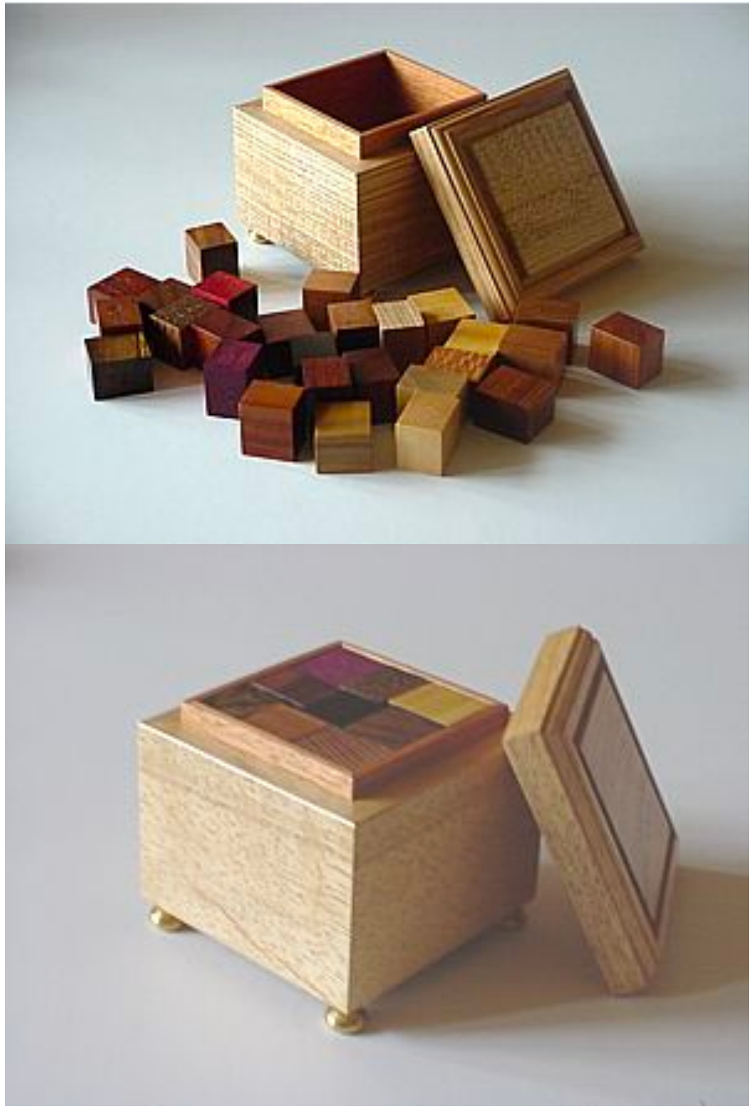
Hoffman's Packing Puzzle

- **Given:** Twenty-seven identical blocks with dimensions $A \times B \times C$ where

$$\frac{A+B+C}{4} < A < B < C.$$

- **Goal:** Pack all blocks into a box with sides $A+B+C$.

  $(e.g.,\ A = 4, B = 5, C = 6.)$



Packing Problems

- Goal: Pack some items under some constraints.
- Example: Bin Packing, Knapsack – two classical NP-hard problems.

Packing Problems

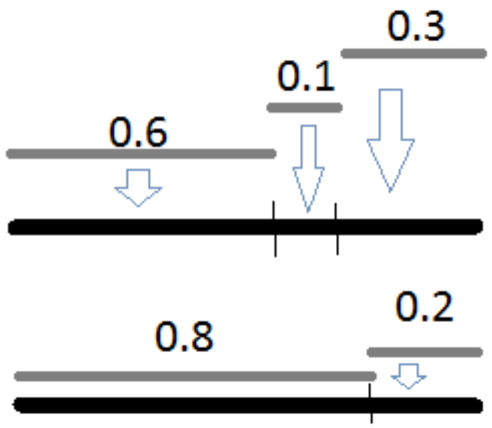
- Studied in computer science and optimization from 1960's. [Gilmore-Gomory, Operations Research '61]
- Among Karp's 21 NP-complete problems.
- The cornerstone of approximation algorithms.
- The term approximation algorithms was first coined for near-optimal bin packing algorithms [Johnson, STOC '73].
- Knapsack and Bin Packing has most needed implementations among all NP-hard problems [*Market Research* by Skiena, '99].

Bin Packing Problem

- Given : n items with sizes $s_1, s_2, \ldots s_n$, s.t. $s_i \in (0,1]$,
- Goal: Pack all items into min # of unit bins.
- Example: items {0.8, 0.6, 0.3, 0.2, 0.1} can be packed in 2 unit bins: {0.8, 0.2} and {0.6, 0.3, 0.1}.
- 3/2 hardness of approximation (from *Partition*).

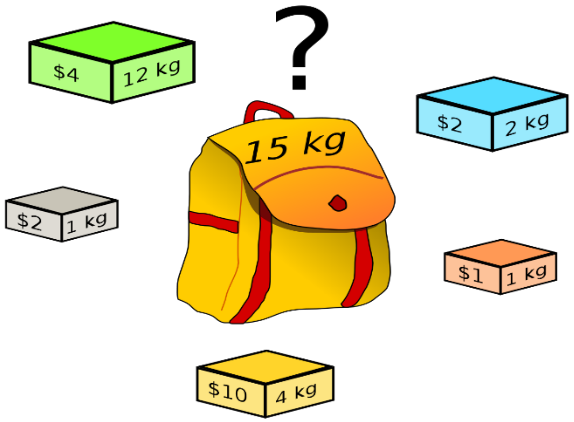
\- This does not rule out OPT+1 guarantee.

- delaVega-Lueker, Combinatorica ’81: APTAS,
- Karp-Karmarkar, FOCS ’82: $\mathrm{OPT} + O(\log^2(OPT))$,
- Hoberg-Rothvoss, SODA ’17: $\mathrm{OPT} + O\big(\log(OPT)\big)$.

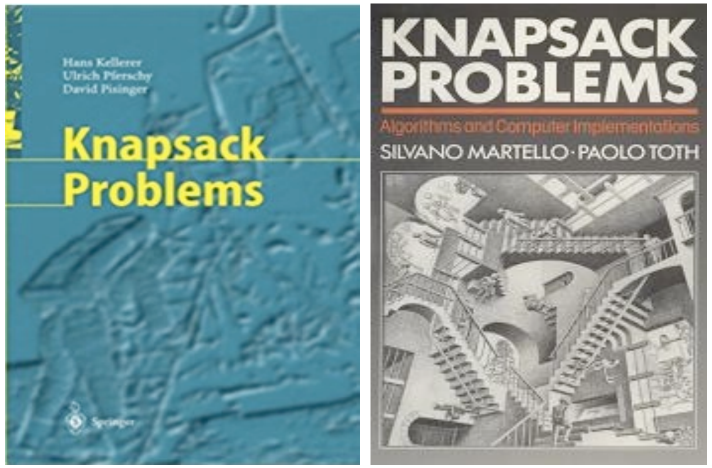


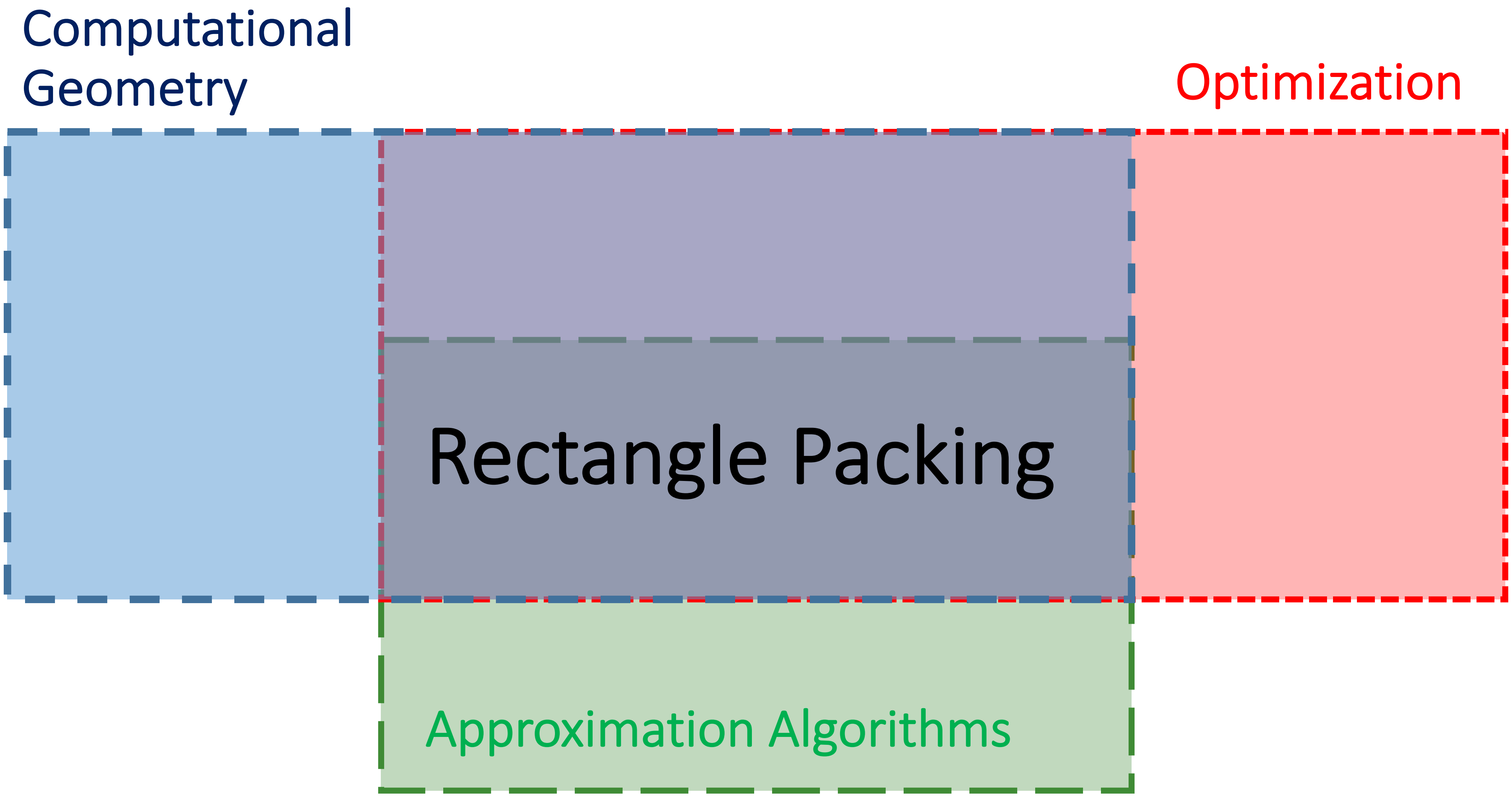
Knapsack Problem

- Given: I, a set of n items where item i has profit $p_i \in \mathbb{Z}$ and size $s_i \in \mathbb{Z}$; Knapsack of size $W \in \mathbb{Z}$.
- Goal: Find the maximum profit subset $S \subseteq I$ that can be packed in the knapsack, i.e., the total size of items in S is at most W.



- Weakly NP-hard: Admits pseudo-polynomial time exact algorithm.
- FPTAS: in $O\left(n \log\frac{1}{\epsilon} + \frac{1}{\epsilon^4}\right)$-time. [Lawler FOCS '77].




Computational
Geometry
Optimization
Rectangle Packing
Approximation Algorithms

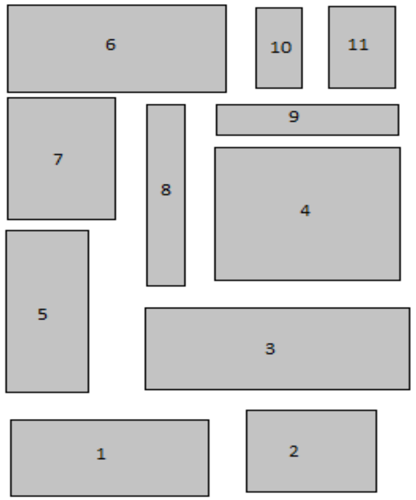
1. 2-D Geometric Bin Packing

2-D Geometric Bin Packing

- Given: Collection of rectangles (by width, height),
- Goal: Pack them into minimum number of unit square bins.

\- Orthogonal Packing: rectangles packed parallel to bin edges.

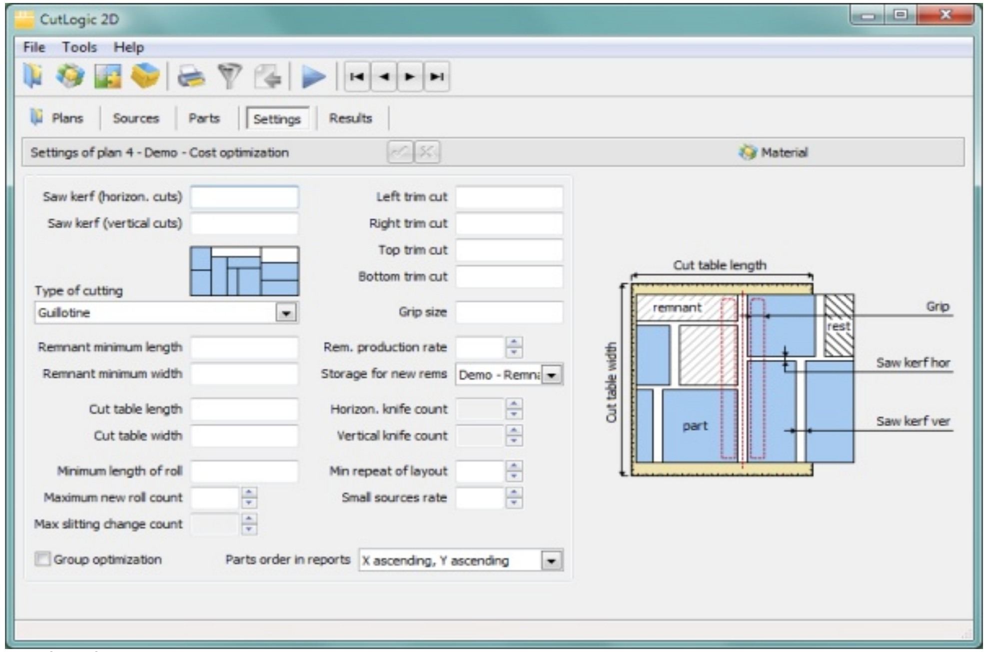
\- With 90 degree *rotations* and *without rotations*.



Applications:

- Cloth cutting, steel cutting, wood cutting
- Placing ads in newspapers
- Memory allocation in paging systems

- Truck Loading
- Palletization by robots



2D BP: Tale of approximability

Algorithm (Asymptotic)
2.125 [Chung Garey Johnson, JACM '82]
$2+\epsilon$ [Kenyon-Remilla, FOCS'96]
1.69 [Caprara, FOCS'02]
1.52 [Bansal-Caprara-Sviridenko, FOCS'06]
1.5 [Jansen-Praedel, SODA'13]
1.405 [Bansal-K., SODA'14] (with and w/o rotations)

Hardness
No APTAS (from 3D Matching) [Bansal-Sviridenko SODA'04],
3793/3792 (with rotation), 2197/2196 (w/o rotation) [Chlebik-Chlebikova '09]

- d-dimensional (d>2) geometric bin packing: 1.69^{d-1} [Caprara, FOCS'02].
- APTAS for d-dimensional squares: [Bansal-Sviridenko, SODA'04].

Configuration LP

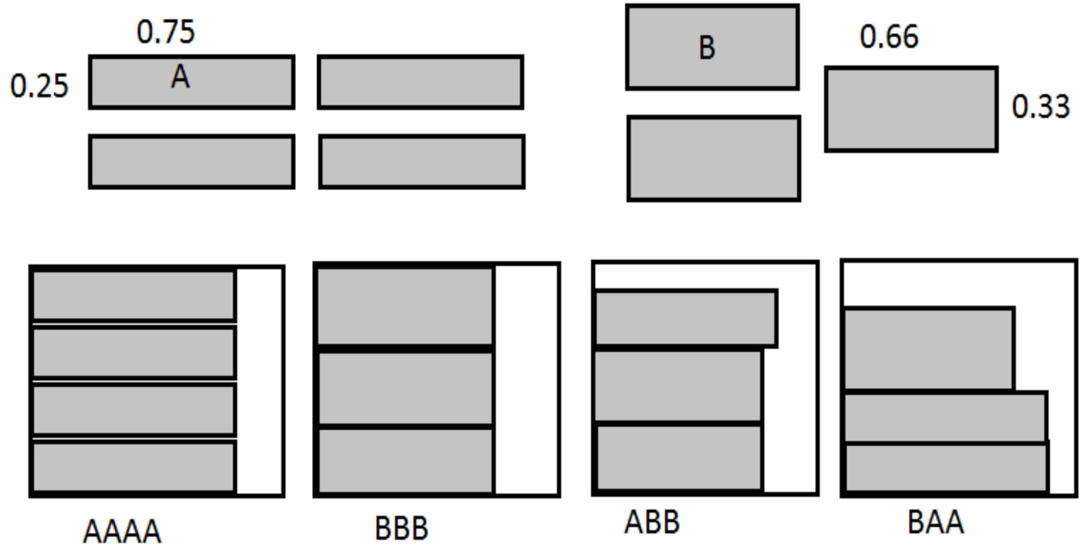
- $\mathbb{C}$*: set of configurations(possible way of feasibly packing a bin).*

Primal:

$$\min \{\sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \; (i \in I), x_C \geq 0 \; (C \in \mathbb{C})\}$$

Dual:

$$\max \{\sum_{i \in I} v_i : \sum_{i \in C} v_i \leq 1 \; (C \in \mathbb{C}), v_i \geq 0 (i \in I)\}$$



- Problem: Exponential number of configurations!
- Solution: Can be solved within $(1+\epsilon)$ accuracy using separation problem for the dual.

Configuration LP

- $\mathbb{C}$*: set of configurations(possible way of feasibly packing a bin).*

Primal:

$$\min \{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \; (i \in I), x_C \geq 0 \; (C \in \mathbb{C}) \}$$

Dual:

$$\max \{ \sum_{i \in I} v_i : \sum_{i \in C} v_i \leq 1 \; (C \in \mathbb{C}), v_i \geq 0 (i \in I) \}$$

Dual Separation problem ⇒
2-D Geometric Knapsack:
Given one bin, pack as
much area as possible.
- PTAS [BCJPS ISAAC 2009]

- Problem: Exponential number of configurations!
- Solution: Can be solved within $(1 + \epsilon)$ accuracy using separation problem for the dual.

Round and Approx (R&A) Framework [Bansal-K. '14]

- Given a packing problem Π

1. If the configuration LP is solved within $(1+\epsilon)$ factor

$$\min \{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \ (i \in I), x_C \geq 0 \ (C \in \mathbb{C}) \}$$

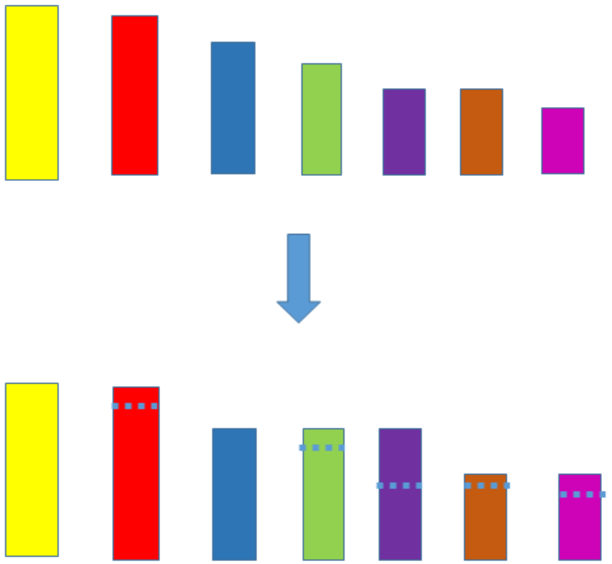
2. There is a ρ approximation rounding-based algorithm.

- Then there is (1+ln ρ) approximation for Π.

Rounding based Algorithms:

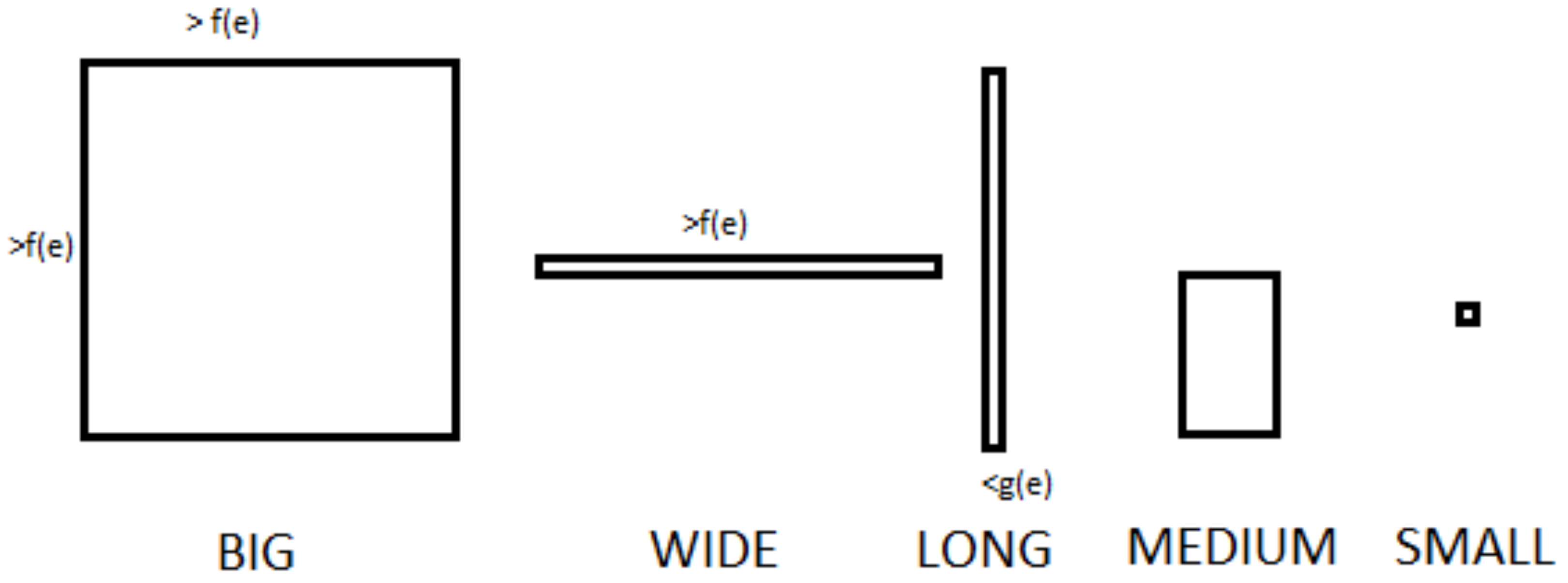
- Rounding based algorithms are ubiquitous in bin packing.
- Items are replaced by slightly larger items from O(1) types.
- Loss:
  Due to larger items.
- Gain:

  Fewer configurations. O(1) types of large items imply rounded instance can be solved optimally.
- Example: Linear grouping [delaVega-Luker, Kenyon-Remilla], Geometric Grouping [Karp-Karmarkar], Harmonic Rounding [Lee-Lee, Caprara, Bansal et al.], JP rounding [JansenPradel].



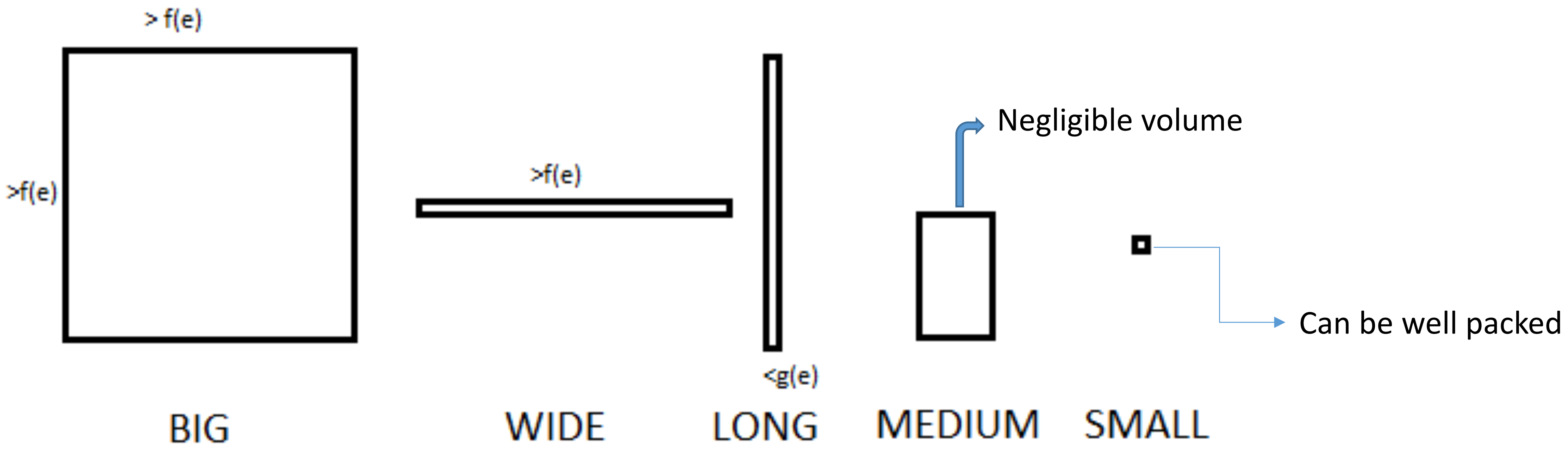
Rounding based Algorithms in 2D

- Classification of items into big, wide, long, medium and small by defining two parameters $f(\epsilon)$ and $g(\epsilon)(\ll f(\epsilon))$ such that total volume of medium rectangles is $\epsilon.Area(I)$.



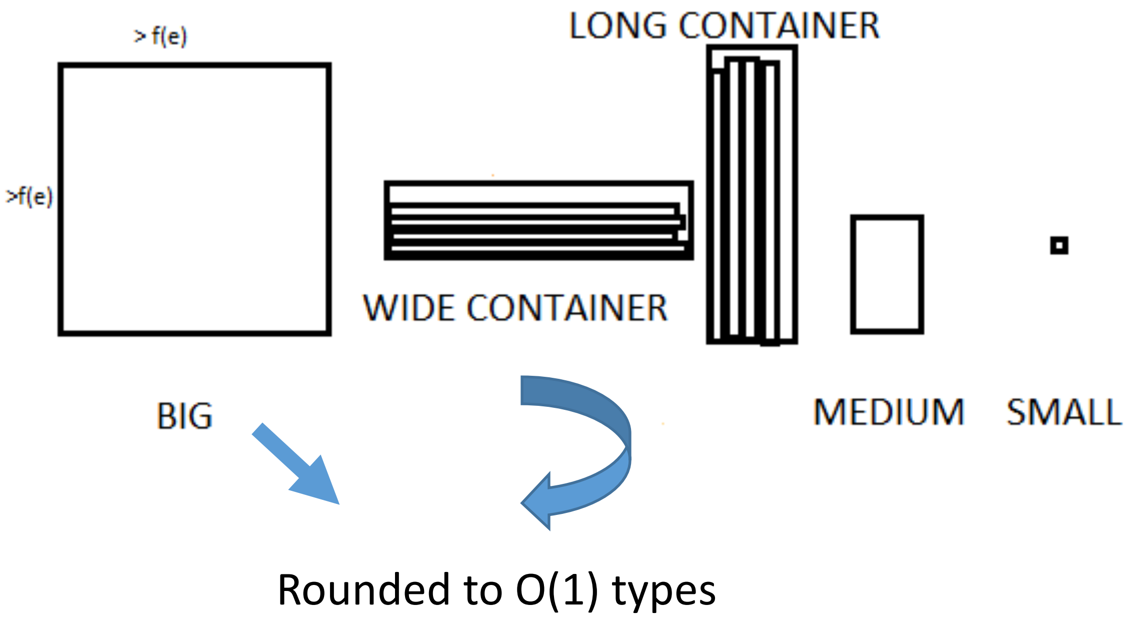
Rounding based Algorithms in 2D

- Classification of items into big, wide, long, medium and small by defining two parameters $f(\epsilon)\ and\ g(\epsilon)(\ll f(\epsilon))$ such that total volume of medium rectangles is $\epsilon.Area(I)$.



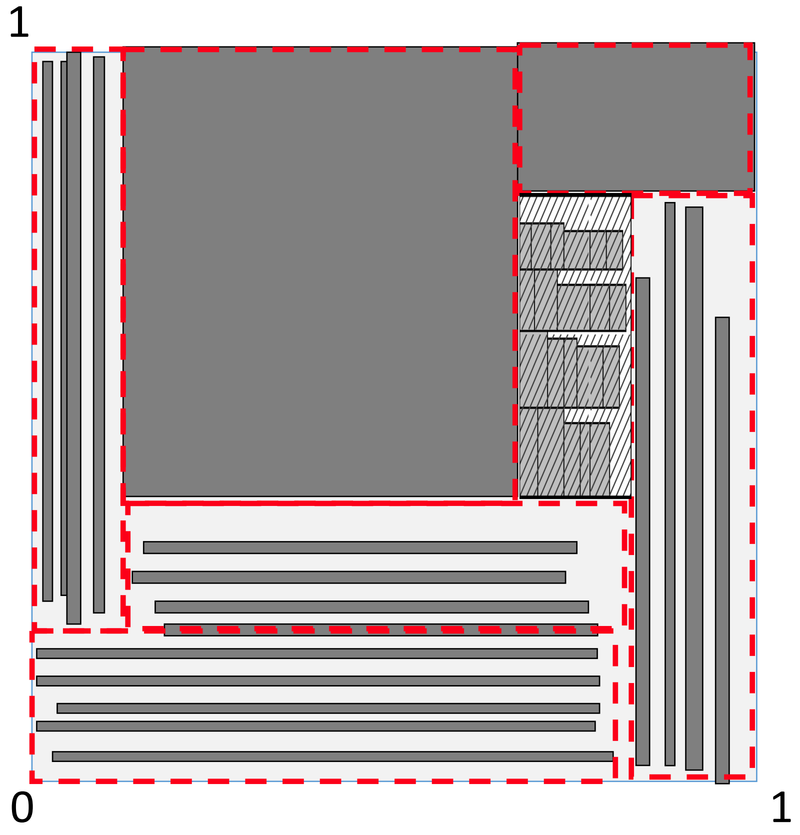
Rounding based Algorithms

- Skewed (wide/long) items are packed into *containers*.
  (i) it has large size in each dimensions and
  (ii) items are packed into containers with a negligible loss of volume.
- Containers and big items are rounded to O(1) types so that we can find near-optimal packing of big items and containers in polynomial time.



Rounding based Algorithms

- Each item is packed in O(1)-type of containers.
- Existence of such packing implies that constructively we can find it.



Round and Approx Framework (R & A)

- 1. Solve configuration LP using APTAS. Let $z^* = \sum_{\{C \in \mathbb{C}\}} x_C^*$.

Primal:

$$\min \left\{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \; (\, i \in I), x_C \geq 0 \; (C \in \mathbb{C}) \right\}$$

Round and Approx Framework (R & A)

- 1. Solve configuration LP using APTAS. Let $z^* = \sum_{\{C \in \mathbb{C}\}} x_C^*$.
- 2. Randomized Rounding: For q iterations :

  select a configuration C' at random with probability $\frac{x_{C'}^*}{z^*}$.

Primal:

$$\min \{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \ (\, i \in I), x_C \geq 0 \ (C \in \mathbb{C}) \, \}$$

Round and Approx Framework (R & A)

- 1. Solve configuration LP using APTAS. Let $z^* = \sum_{\{C \in \mathbb{C}\}} x^*_C$.
- 2. Randomized Rounding: For q iterations :

  select a configuration C' at random with probability $\frac{x^*_{C'}}{z^*}$.
- 3. Approx: Apply a ρ approximation rounding based algorithm A on the residual instance S.
- 4. Combine: the solutions from step 2 and 3.

R & A Rounding Based Algorithms

- Probability item i left uncovered after rand. rounding

$= \left(1 - \sum_{\{C \ni i\}} \frac{x_C^*}{z^*}\right)^q \leq \frac{1}{\rho}$ by choosing $q = (\ln \rho) LP(I)$.

- Number of items of each type shrinks by a factor ρ

  e.g., $\mathbb{E}\left[|B_j \cap S|\right] = \frac{|B_j|}{\rho}$.for some item type B_j.

- Concentration using Independent Bounded Difference Inequality.

Proof Sketch

- Rounding based Algo : O(1) types of items $= O(1)$ number of constraints in configuration LP.
- $ALGO(S) \approx OPT(\tilde{S}) \approx LP(\tilde{S})$.
- As # items for each item type shrinks by ρ, $LP(\tilde{S}) \approx \frac{1+\epsilon}{\rho} LP(\tilde{I})$.
- ρ − approximation: $ALGO(I) \approx LP(\tilde{I}) \leq \rho\, OPT(I) + O(1)$.
- $ALGO(S) \approx OPT(I)$.

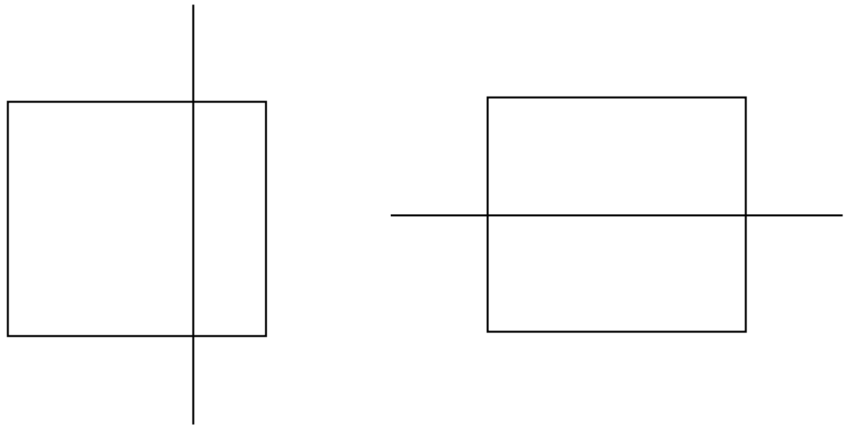
Proof Sketch

- Thm: R&A gives a $(1 + \ln\rho + \epsilon)$ approximation.
- Proof:
- Randomized Rounding : $q = \ln\rho . LP(I)$
- Residual Instance S $= (1 + \epsilon)OPT(I) + O(1)$.

- Round + Approx => $(\ln\rho + 1 + \epsilon)OPT(I) + O(1)$.

Guillotine Packing

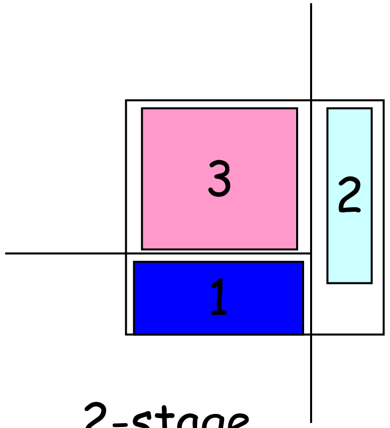
Guillotine Cut: Edge to Edge cut across a bin



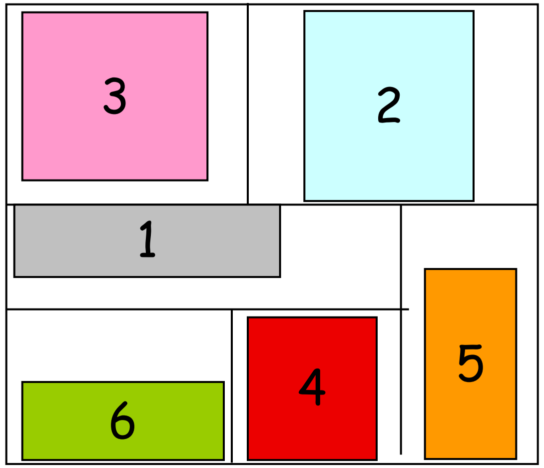
Objective: Minimize number of bins such that packing in each bin is a guillotine packing.

Guillotine Packing => General Bin packing

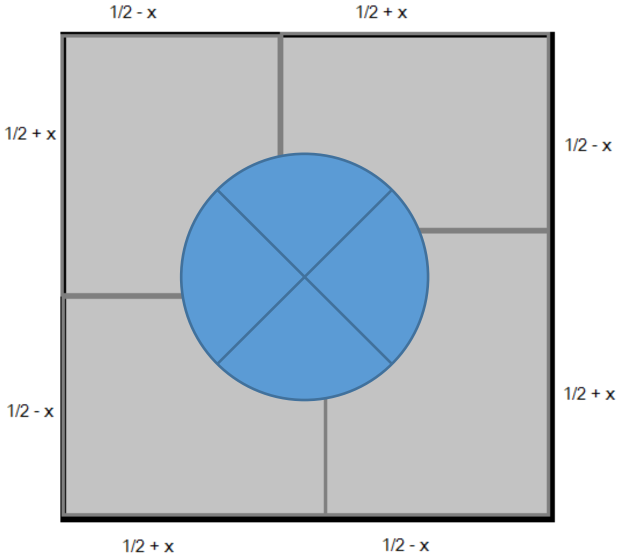
Guillotine cut: edge to edge cut across a bin



2-stage



4-stage

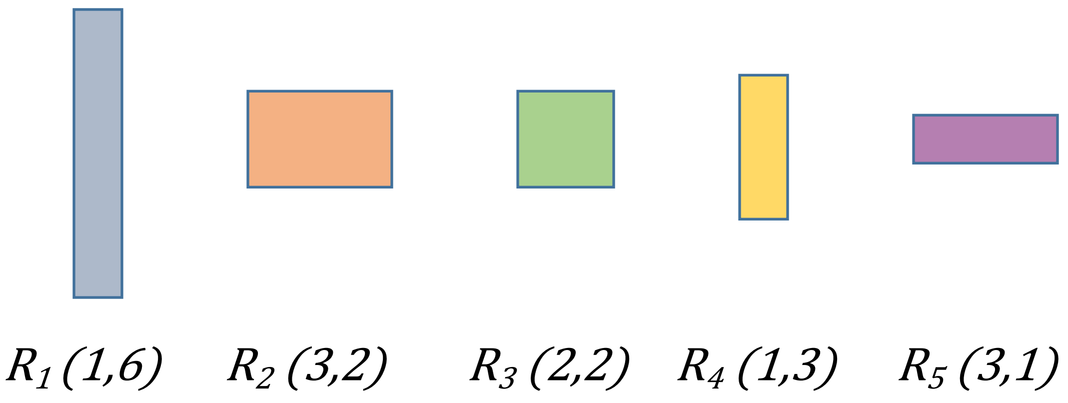


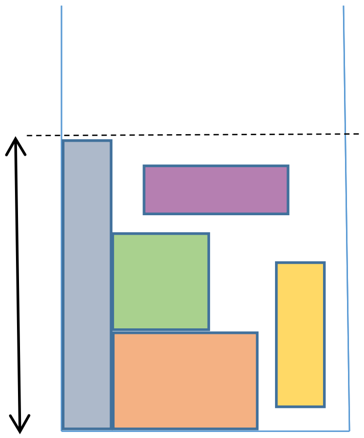
- APTAS for guillotine 2-D bin packing [Bansal Lodi Sviridenko, FOCS'05].
- Conjecture: Given any packing of m bins, there is a guillotine packing in $4m/3$ bins. This will imply $\left(\frac{4}{3}+\varepsilon\right)$-approximation for 2-D BP.

2. Strip Packing

Strip Packing Problem: (2-D)

- Input :
  - Rectangles $R_1, R_2, \ldots, R_n$; Each R_i has integral width and height (w_i, h_i).
  - A strip of integral width W and infinite height.
- Goal :
  - Pack all rectangles minimizing the height of the strip.
  - Axis-parallel non-overlapping packing.





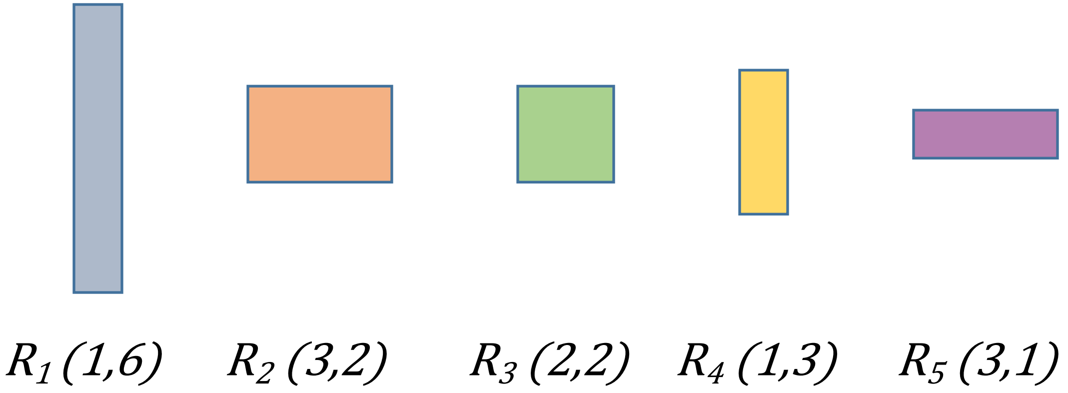
Variant 1:

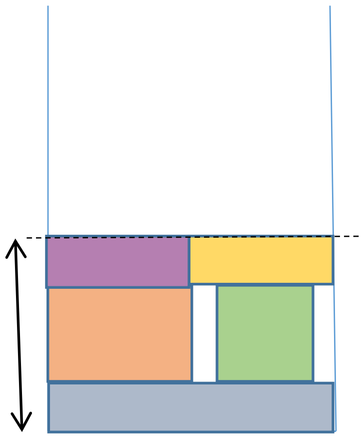
No rotations are allowed!

Strip Packing Problem: (2-D)

- **Input :**
  - Rectangles $R_1, R_2, \ldots, R_n$; Each R_i has integral width and height (w_i, h_i).
  - A strip of integral width W and infinite height.

- **Goal :**
  - Pack all rectangles minimizing the height of the strip.
  - Axis-parallel non-overlapping packing.



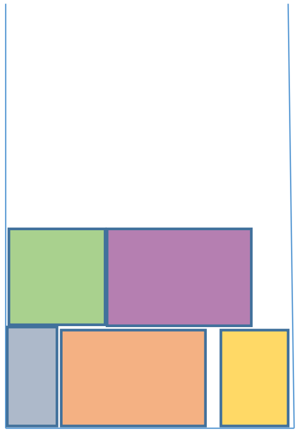


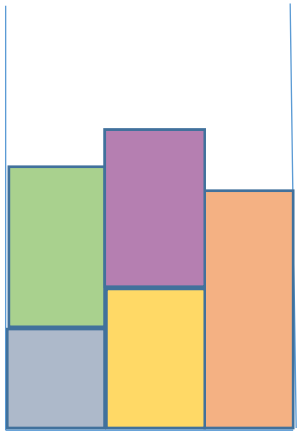
Variant 2:

90º rotations are allowed!

Strip Packing:

- Strip Packing generalizes
  - bin packing (when all rectangles have same height),
  - makespan minimization (when all rectangles have same width).





Tale of approximability.

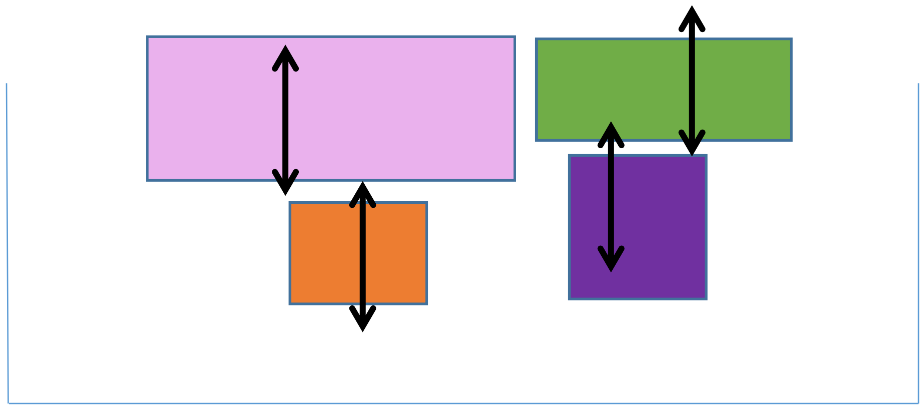
- Asymptotic Approximaton:
- Asymptotic PTAS [Kenyon-Remila, FOCS'96] (Without rotations),
- Asymptotic PTAS [Jansen-vanStee, STOC'05] (With rotations).
- Absolute Approximation:
- 2.7-appx. [First-Fit-Decreasing-Height, Coffman-Garey-Johnson-Tarjan '80].
- 5/3+ε [Harren-Jansen-Pradel-vanStee, Comp.Geom.'14].
- Hardness of appx in poly-time: 3/2 (from Bin Packing).
- Hardness of appx in pseudo poly-time: 5/4 (from 3-partition).
- Pseudo-polytime: (5/4+ε)-appx [Jansen-Rau ESA'19].

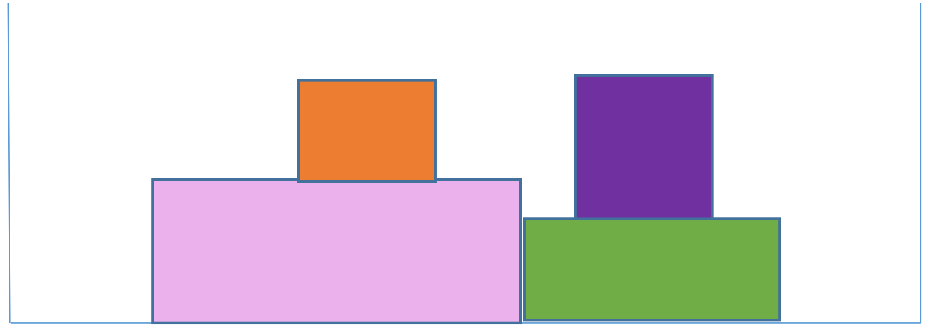
3. Dynamic Storage Allocation (DSA)

DSA

- Input: Rectangles $R_1, R_2, ..., R_n$; Each R_i has width w_i, height h_i, and fixed starting position on x-coordinate x_i;
  - A strip of integral width 1 and infinite height.

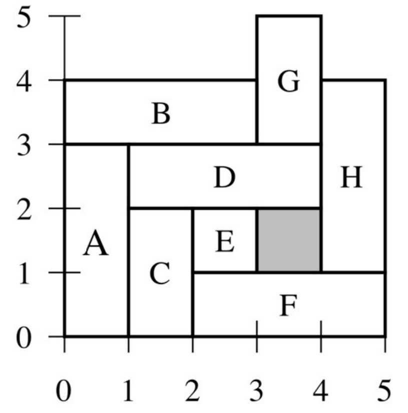
- Goal: Pack (non-overlapping and axis-parallel) all rectangles into the strip of minimum height by sliding the rectangles vertically but not horizontally.





DSA

- Important applications in contiguous resource allocation (e.g., memory, bandwidth)
- Generalizes interval coloring (when items have same height),
- NP-hard [Stockmeyer '76], even for squares.
- Possibility of PTAS is open.
- h_{max}:=maximum height rectangle, $LOAD$:= maximum sum of heights of rectangles that intersect any vertical line. Then, $OPT = LOAD(1 + O\left(\frac{h_{max}}{L}\right)^{1/7})$.
- $(2+\varepsilon)$-appx, even for squares [Buchsbaum et al., STOC'03],
- If we can drop ε-fraction of items, we can achieve a packing in height $(1+\varepsilon)\ OPT$ [Momke et al., '20].



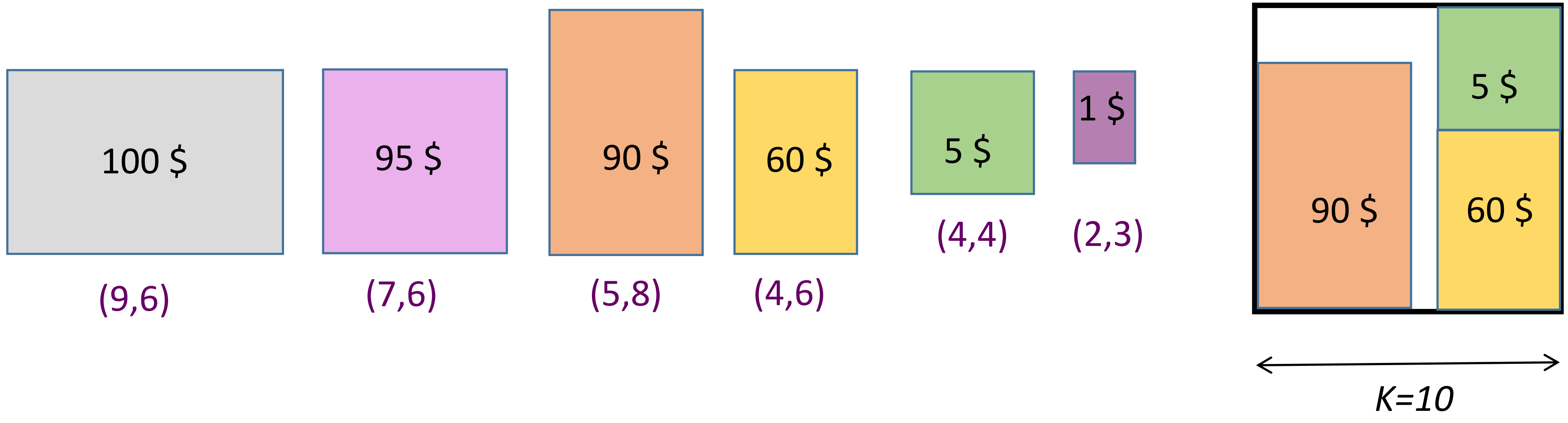
h_{max}=3,
$LOAD$=4, OPT=5

4. 2-D Geometric Knapsack (2-D GK)

Geometric Knapsack: (2-D)

- Input :
  - Rectangles $I := \{R_1, R_2, \ldots, R_n\}$; Each R_i has integral width and height (w_i, h_i) and profit p_i.
  - A Square $K \times K$ knapsack.

- Goal : Find an axis-parallel non-overlapping packing of a subset of input rectangles into the knapsack that maximizes the total profit.



Variant 1: 2DK

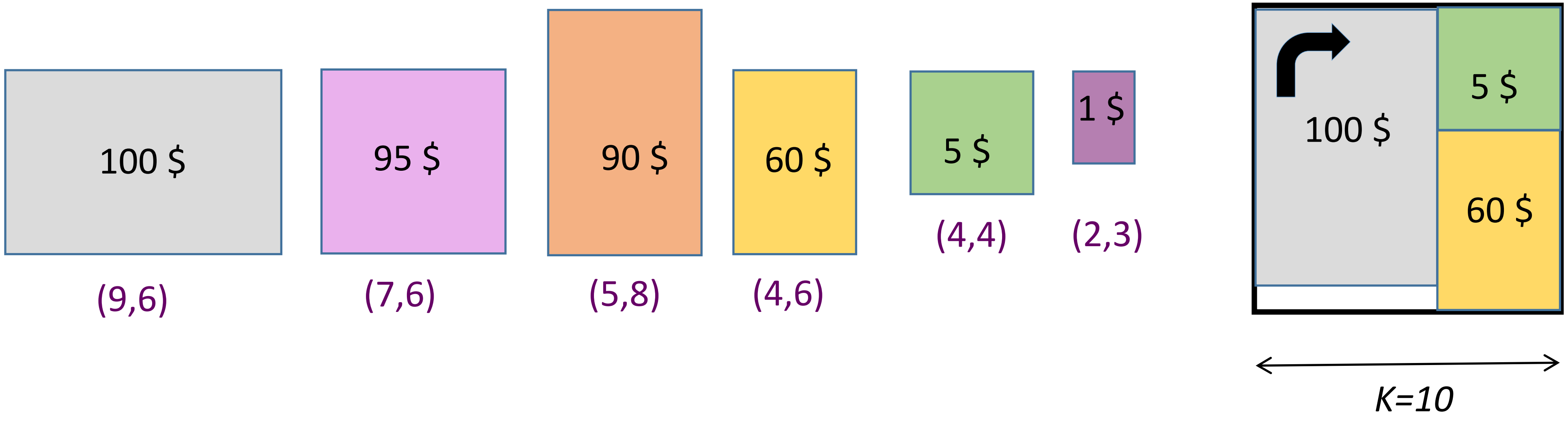
No rotations are allowed!

OPT=155

Geometric Knapsack: (2-D)

- Input :
  - Rectangles $I := \{R_1, R_2, \ldots, R_n\}$; Each R_i has integral width and height (w_i, h_i) and profit p_i.
  - A Square $K \times K$ knapsack.

- Goal : Find an axis-parallel non-overlapping packing of a subset of input rectangles into the knapsack that maximizes the total profit.



Variant 2: (2DKR)

90 degree rotations are allowed!

OPT=165

Geometric Knapsack: Complexity

- Geometric Knapsack is Strongly NP-hard
  (even when all items are squares with profit 1), [Leung et al., 1990]
  - Remains NP-hard even if the input is given in unary.
  - No exact algorithm even in pseudo-polynomial time (unless P=NP).
- Not known whether the problem is APX-hard. So, the existence of a PTAS/QPTAS/PPTAS is still open!
- (1+ε)-approximation known if
  - profit of an item is equal to its area. [Bansal et al., ISAAC '09].
  - items are relatively small [Fishkin et al., MFCS '05].
  - items are squares [Wiese-Heydrich, SODA '17].

Geometric Knapsack:

- (2+ε)-approximation [Jansen-Zhang, SODA'04]

  - for both with and without rotations.
  - even in the cardinality case (when all profits are 1).

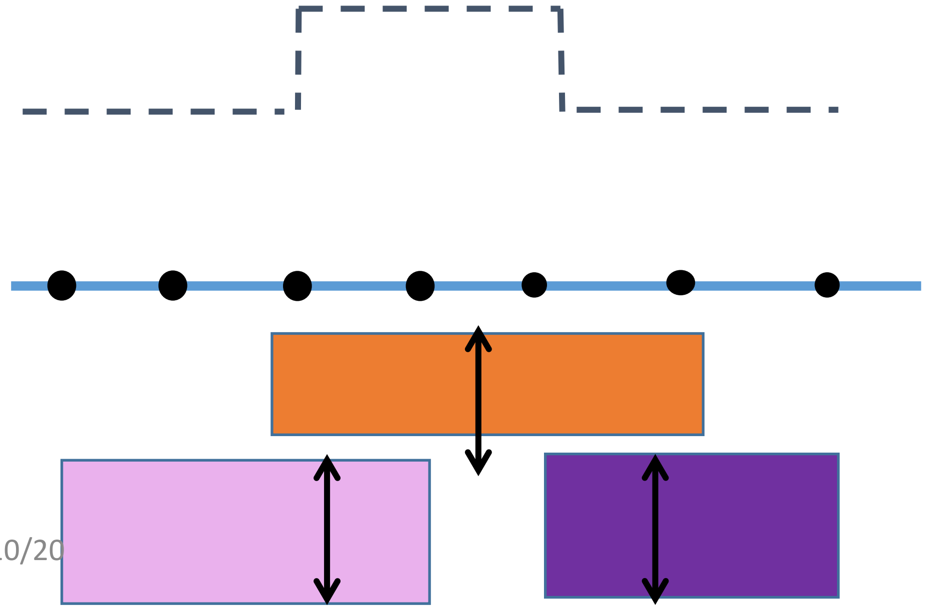
- Broke the barrier of 2 [Galvez-Grandoni-Ingala-K.-Wiese, FOCS'17]

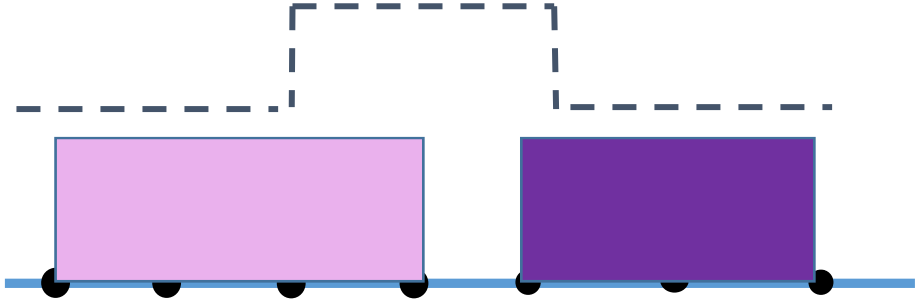
  - Without rotations: $(17/9+\varepsilon)<1.89$-appx.
  - With rotations: $(1.5+\varepsilon)$-appx.
  - Cardinality case: 1.72, $(4/3+\varepsilon)$-appx., resp.

5. Storage Allocation Problem (SAP)

SAP

- Input: A path with edge capacities and a set of tasks (rectangles) that are specified by start and end vertices (fixed starting coordinate and width), demands (heights) and profits.
- Goal: Select a subset of tasks that can be drawn as non-overlapping rectangles underneath the capacity profile.





SAP: Tale of approximability

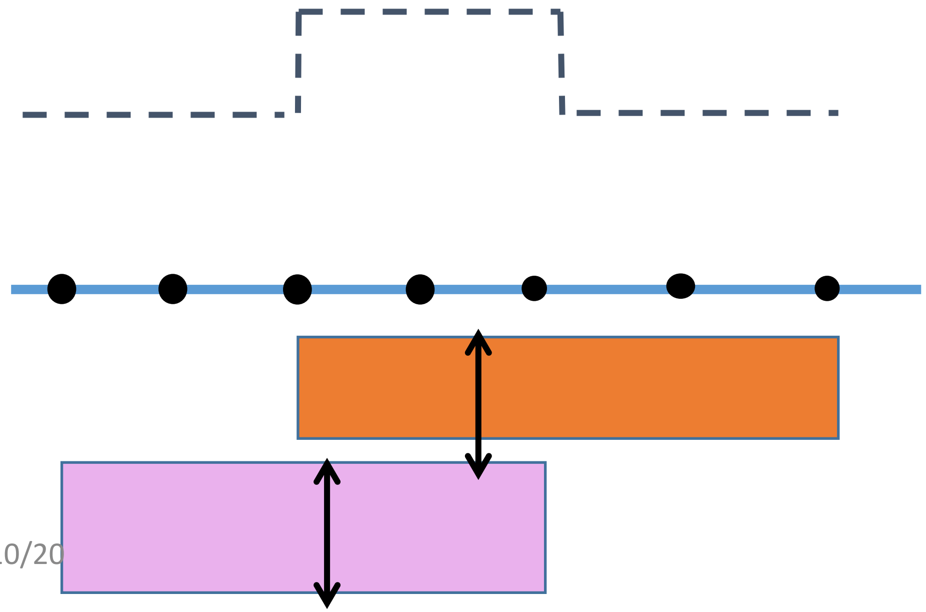
- Generalizes knapsack.
- Special case: Uniform-SAP (when all edges have same capacity)
  – 7-appx [Bar-Noy et al, STOC'00].
- General case: $(9 + \varepsilon)$-approximation [Bar-Yehuda et al, SPAA'13].
- $(2 + \varepsilon)$-approximation [Momke-Wiese, ICALP'15].
- Uniform-SAP: 1.969 [Momke-Wiese, '20].
- General-SAP: QPTAS with resource augmentation. [Momke-Wiese, '20].

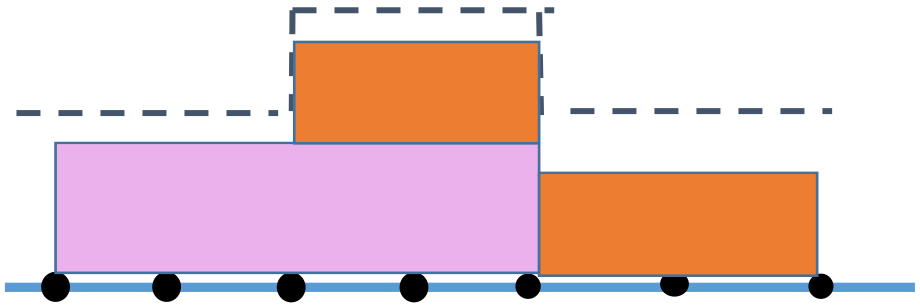
6. Unsplittable Flow on a Path (UFP)

UFP (sliced version of SAP)

- Input: A path with edge capacities and a set of tasks (rectangles) that are specified by start and end vertices (fixed starting coordinate and width), demands (heights) and profits.

- Goal: Select a subset of tasks such that total demand of selected tasks at any edge is less than the edge capacity.





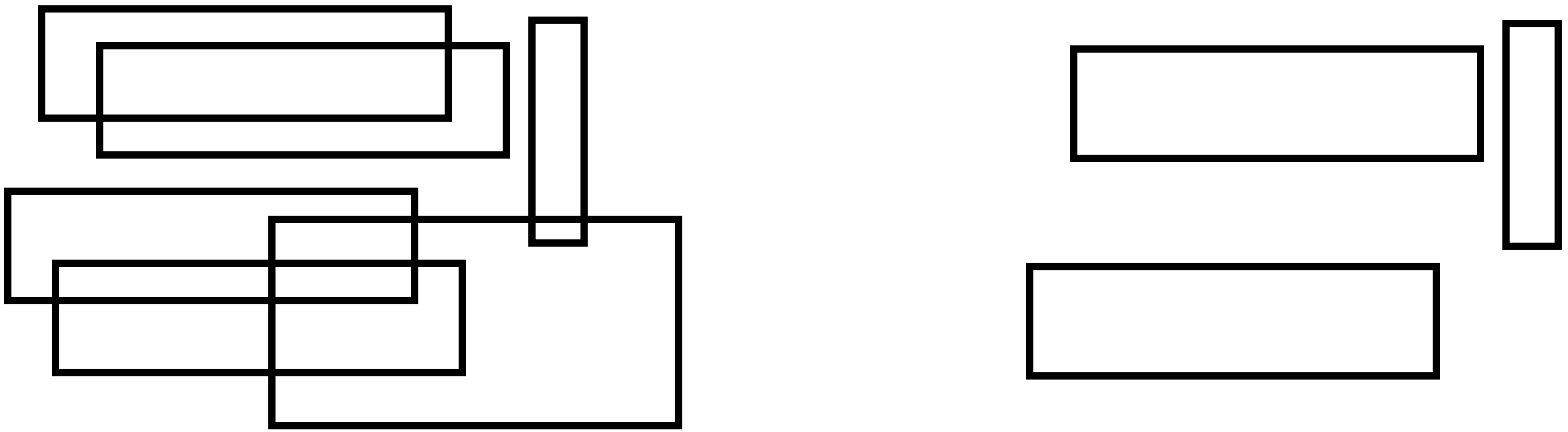
UFP: A tale of approximability

- Strongly NP-hard, even for uniform edge capacities and uniform profits.
- QPTAS [Bansal et al, STOC'06, Batra et al, SODA'15] ,
- O(log n)-apprx. [Bansal et al, SODA'09],
- $(7 + \varepsilon)$-appx [Bonsma et al, FOCS'11],
- $(2 + \varepsilon)$-appx [Anagnostopoulos et al, SODA'14],
- $(\frac{5}{3} + \varepsilon)$-appx [Grandoni et al. STOC'18],
- Possibility of PTAS is still open!

7. Maximum Weighted Independent Set of Rectangles (MWISR)

MWISR

- Input: n axis-parallel rectangles (each with associated profit) on a plane.
- Goal: Find maximum profit subset of disjoint rectangles.
- Special case: uniform profit (MISR).
- Applications: data-mining, map-labeling, etc.



MWISR: tale of approximability

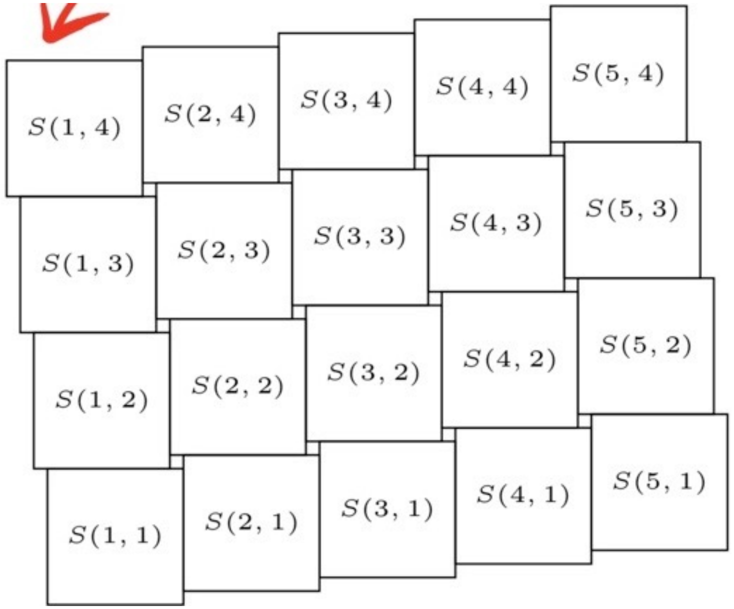
- NP-hard.
- Folklore: O(log n)
- MISR: O(log log n)-approximation [Chalermsook-Chuzhoy, SODA'09]
- MWISR:
- O(log n/log log n) [Chan-HarPeled, SoCG'09]
- PTAS for pseudodiscs (e.g. squares) [Chan-HarPeled, SoCG'09]
- $(1+\varepsilon)$-appx in $n^{poly(\log n)}$ [Adamaszek-Wiese, FOCS'13]
- $(1+\varepsilon)$-appx in $n^{poly(\log\log n)}$ [Chuzhoy-Ene, STOC'16]
- PTAS, even O(1)-appx is open!

Pach-Tardos Conjecture

Pach-Tardos Conjecture

- Conjecture: For any set of n non-overlapping axis-parallel rectangles there is a guillotine cutting sequence with only axis-parallel cuts separating $\Omega(n)$ of them.

- Known upper bound: n/2 (also for squares).
- Known lower bound: n/log n.
- The conjecture is true for squares! [Abed et al, APPROX'15]
- Theorem: [Abed et al, APPROX'15] If the conjecture is true, then there is a $O(1)$-approximation algorithm for MISR with running time $O(n^5)$.



Other related problems.

- Round-UFP, Round-SAP, coloring of rectangles.
- Min-area rectangle packing: APTAS [Bansal-Sviridenko, SODA'04]
- Circle and other geometric objects.
- Vector: when items are multidimensional vectors.
  d-dim vector bin packing: $0.81 + O(\log d)$ [Bansal-K.-Elias, SODA'16]
- Graph: weighted biparitite edge coloring, weighted biparitite matching, generalized assignment problem. [K.-Singh, FSTTCS'15]
- Approximation and Online Algorithms for Multidimensional Bin Packing: A Survey, Christensen-K.-Pokutta-Tetali, Computer Science Review 2017.

Summary of present status

- Though these variants are related, their approximability and techniques are quite diverse, e.g.,

  -- Strip packing: admits APTAS,

  -- Independent set of rectangles:

  admits QPTAS and $(\log n / \log\log n)$-polytime approximation.

  -- Geometric knapsack:

  may or may not admit PTAS/QPTAS/PPTAS, < 2- appx. Known.

  -- SAP/DSA: APX-hardness not known, Barrier of 2-approximation.

  -- d-dim vector packing: No APTAS. Best known approx.: $O(\log d)$

  -- d-dim geometric bin packing: No APTAS. Best known approx.: 1.69^{d-1}.

Top 10 open problems

1. Algorithm with OPT+O(1)-guarantee for bin packing.
2. A poly(d)-approximation or hardness for d-dim geometric bin packing?
3. Resolve guillotine conjecture for 2-D bin packing.
4. $(\frac{3}{2} + \varepsilon)$-approximation for strip packing?
5. PTAS (or PPTAS or QPTAS) for 2-D geometric knapsack (even with rotations)?
6. PTAS for unsplittable flow on a path?
7. Break the barrier of 2 for dynamic storage allocation.
8. Break the barrier of 2 for storage allocation problem.
9. Resolve Pach-Tardos conjecture.
10. PTAS for maximum independent set of rectangles?