Fundamentals of Vehicle Dynamics

ME5670

Lecture 1-2

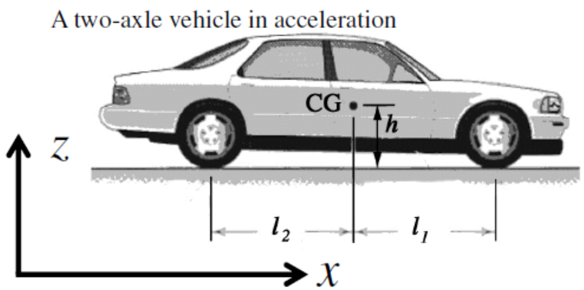
Thomas Gillespie, "Fundamentals of Vehicle Dynamics", SAE, 1992.

http://www.me.utexas.edu/~longoria/VSDC/clog.html

http://www.slideshare.net/NirbhayAgarwal/four-wheel-steering-system

Class timing
Monday: 14:30 Hrs – 16:00 Hrs
Thursday: 16:30 Hrs – 17:30 Hrs

FBD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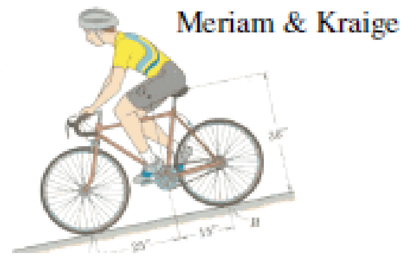
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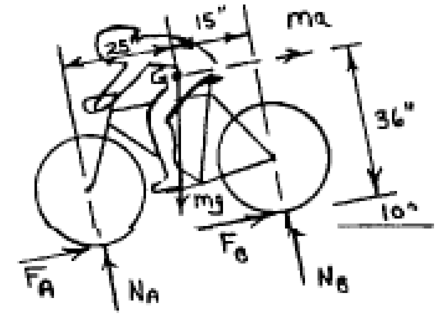
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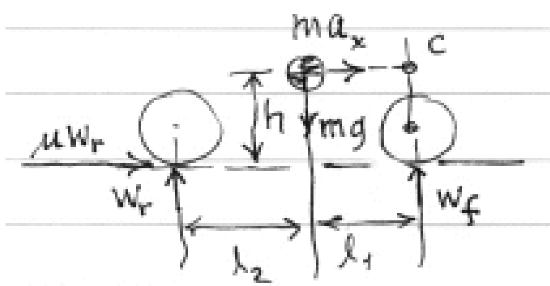
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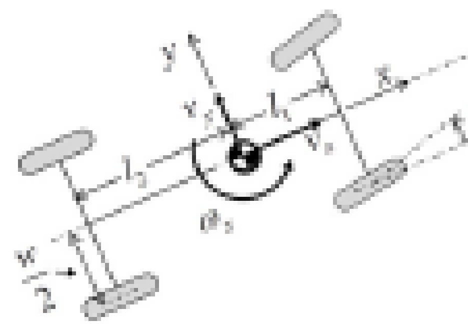

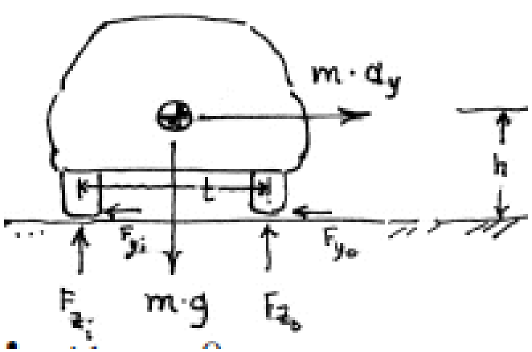
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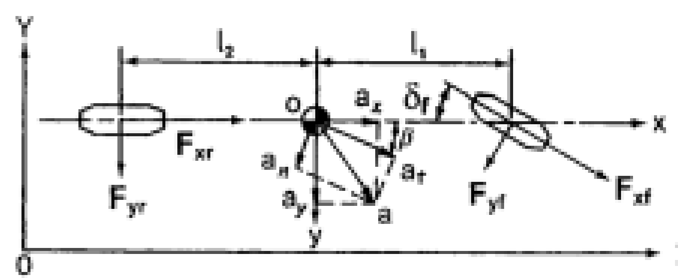
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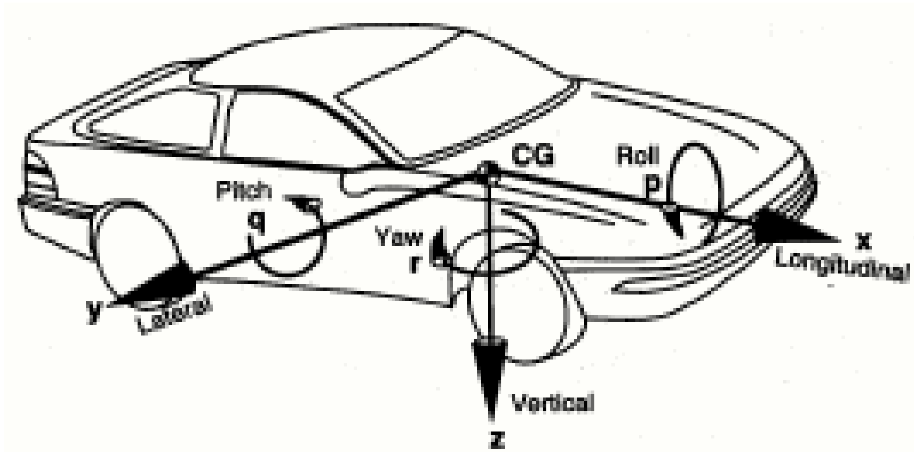
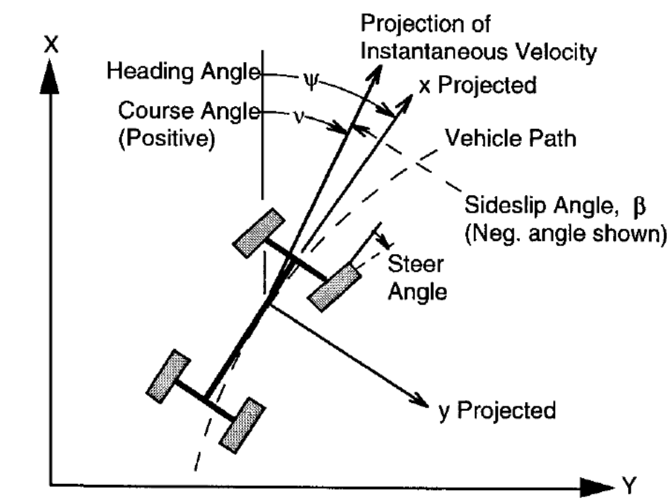
Free Body Diagram



1. Vehicle fixed co-ordinate system:

It is defined with reference to a right-hand orthogonal coordinate system which originates at CG and travels with the vehicle

x – forward
y- lateral
z- downward
p- roll velocity
q- pitch velocity
R – yaw velocity



1. Earth fixed co-ordinate system:

Vehicle altitude and trajectory through the course of a maneuver are defined with respect to a right-hand orthogonal axis system fixed on the earth.

X – Forward travel
Y - Travel to the right
Z - Vertical travel (+ downward)
ψ - Heading angle
ν - Course angle
β – Sideslip angle

Physical Quantities

1. Euler angles:

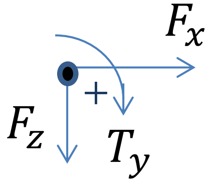
The vehicle fixed co-ordinate system is related to the earth fixed co-ordinate system through the Euler angles.

Euler angles are defined the by the sequence of three angular rotations

- *Beginning with the earth fixed system,*
  the axis system is first rotated about the z axis (yaw)
- *It then rotates about the y-axis (pitch)*
- *Finally, it rotate about the x-axis (roll) to line up with the vehicle fixed co-ordinate system.*

The order of the rotation is strictly adhered to get the resultant altitude

2. Forces and moments:



Forces and moments are normally defined as they act on the vehicle.
The positive sign of longitudinal, vertical and moment in that plane is given by

3. Equilibrium condition:

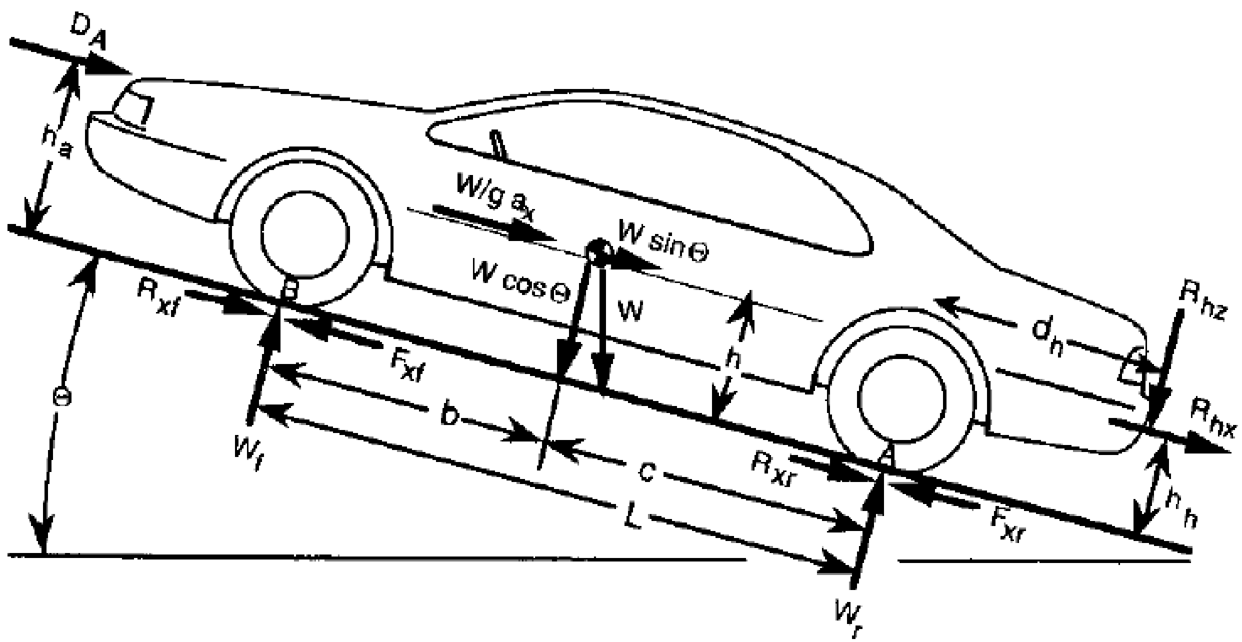
- Translational systems: $\sum F_x = Ma_x; \sum F_z = Ma_z$
- Rotational systems: $\sum T_y = I_{yy}\alpha_y$

Dynamic Axle Loads

1. Dynamic axle loads on a vehicle under arbitrary condition

It is an important step in analysis of acceleration and braking performance because the axle loads determine the tractive effort obtainable at each axle.

- *acceleration*
- *gradeability*
- *maximum speed*



Forces:

W=mg =weight @ C.G.

W_f = *Weight @front wheel*

W_r = *Weight @rear wheel*

F_{xf} =*Traction force at front*

F_{xr} =*Traction force at rear*

R_{xf} =*Rolling resistance at front*

R_{xr} = *Rolling resistance at rear*

D_A = *Aerodynamic load acting on the body at* h_a

R_{hz} =*Vertical load under towing condition*

R_{hx} = *Longitudnal load under towing condition*

Computing Dynamic Axle Loads

Load carried on each axle will consist of a static component, plus load transferred from front to rear due to the other forces acting on the vehicle.

Load on the front axle is found by taking net moment about the point A under the rear tires

Under no acceleration in pitch and taking clockwise direction as positive:

$$W_f L + D_A h_a + \frac{W}{g} a_x h + R_{hx} h_h + R_{hz} d_h + W h \sin\Theta - W c \cos\Theta = 0$$

For uphill altitude: Θ = *+ve*
For downhill altitude: Θ = *-ve*

W_f *can be obtained by solving the above equation*

Similarly, W_r *can be obtained by taking the moment about B under the front wheel.*

$$W_f = (W c \cos\Theta - R_{hx} h_h - R_{hz} d_h - \frac{W}{g} a_x h - D_A h_a - W h \sin\Theta)/L$$

$$W_r = (W b \cos\Theta + R_{hx} h_h + R_{hz} (d_h + L) + \frac{W}{g} a_x h + D_a h_a + W h \sin\Theta)/l$$

Axle Load under Different Conditions

1. Static loads on Level Ground: *When the vehicle sits statically on level ground.*

=> $\Theta=0;\ R_{hx}=0;\ R_{hz}=0;\ a_x=0;\ D_A=0$

Axle loads: $W_{fs} = W\frac{c}{L}$ $\quad W_{rs} = W\frac{b}{L}$

2. Loads on Grades: *The influence of grade on axle loads.*

Grade is defined as the "rise" over the "run".
The ratio of rise over the run is the tangent of the grade angle Θ

The common grades on interstate highways are limited to 4%
On primary and secondary roads, they are limited to 10-12 %

For small grade angle: $\sin\Theta \sim \Theta$ *and* $\cos\Theta \sim 1$

Axle loads:

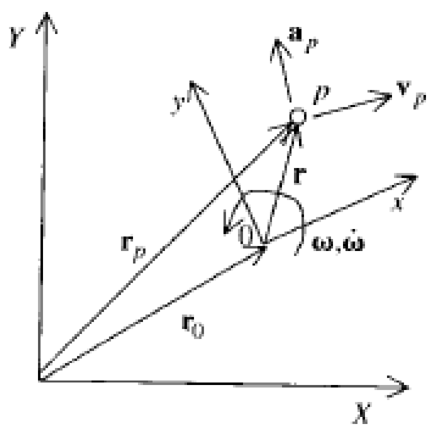
$$W_f = W\left(\frac{c}{L} - \frac{h}{L}\Theta\right) = W_{fs} - W\frac{h}{L}\Theta$$

$$W_r = W\left(\frac{b}{L} + \frac{h}{L}\Theta\right) = W_{rs} + W\frac{h}{L}\Theta$$

Positive grade causes load to be transferred from the front to the rear axle.

Absolute Motion of a Particle

- Consider two frames of reference, a fixed frame XY and a rotating frame Oxy with angular velocity ω
- Let P be a particle moving the plane of the figure and having position vector r w.r.t. both the frames, however, its rate of change will depend on the selected frame of reference.



- Position $$\mathbf{r}_p = \mathbf{r}_0 + \mathbf{r}$$
- Velocity $$\mathbf{v}_p = \dot{\mathbf{r}}_0 + \dot{\mathbf{r}} = \mathbf{v}_0 + \dot{\mathbf{r}}$$

$$\dot{\mathbf{r}} = \mathbf{v}_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{v}_p = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{\text{rel}}$$

- Acceleration $$\mathbf{a}_p = \ddot{\mathbf{r}}_0 + \ddot{\mathbf{r}} = \mathbf{a}_0 + \ddot{\mathbf{r}}$$

$$\ddot{\mathbf{r}} = \dot{\mathbf{v}}_{\text{rel}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}$$

using $$\dot{\mathbf{v}}_{\text{rel}} = \mathbf{a}_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{v}_{\text{rel}}$$

we get

$$\ddot{\mathbf{r}} = \mathbf{a}_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\mathbf{v}_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{r})$$
$$= \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

$$\mathbf{a}_p = \mathbf{a}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

- For any vector A expressed w.r.t. a rotating frame, its absolute change is given by

$$\frac{d\mathbf{A}}{dt} = \left.\frac{\partial \mathbf{A}}{\partial t}\right|_{\text{rel}} + \boldsymbol{\omega} \times \mathbf{A}$$

Example

Example: Karnopp and Margolis P1.11

1.11 Figure P1.11 shows the top view of a vehicle that has mass m and c.g. moment of inertia about the axis out of the page, I_g. The center of gravity is located a distance a from the front axle and a distance b from the rear axle. The half-width of the vehicle is $w/2$. The front wheels can be steered, indicated by the steer angle δ. A body-fixed coordinate frame is attached to the vehicle at its center of gravity and aligned as shown. The body-fixed velocity components of the center of gravity and the yaw angular velocity are indicated.

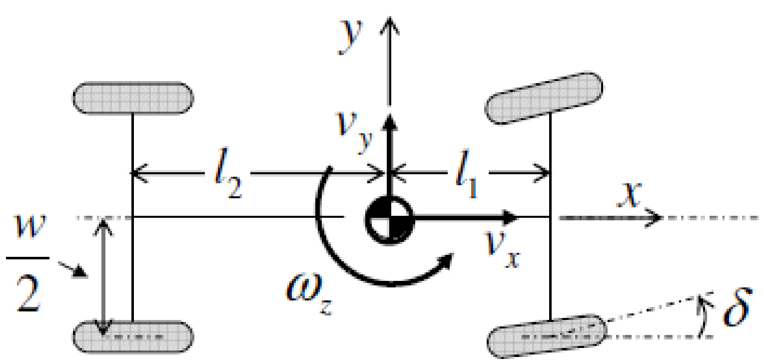
(a) Using arrows and symbols, transfer the c.g. velocity to body-fixed directions at the four wheels.

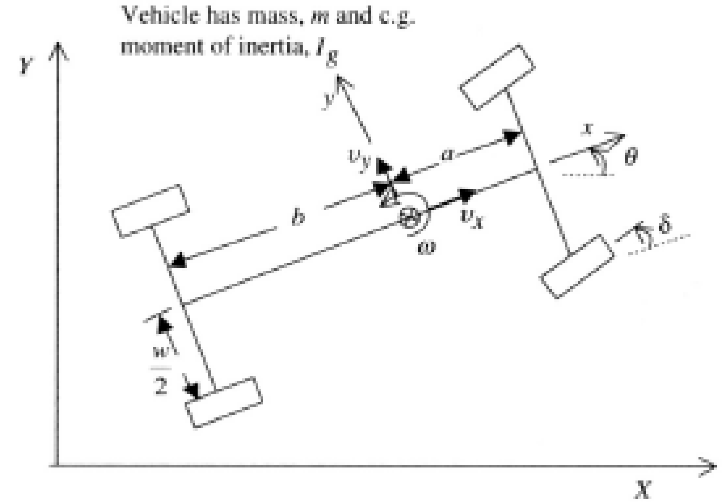
(b) If each wheel is constrained to have no velocity perpendicular to the plane of the wheel, state the kinematic constraints for each wheel.

(cf. Karnopp & Margolis, eqs. 1.18)

$$\vec{V}_p = \vec{V}_o + \vec{\Omega} \times \vec{R}$$

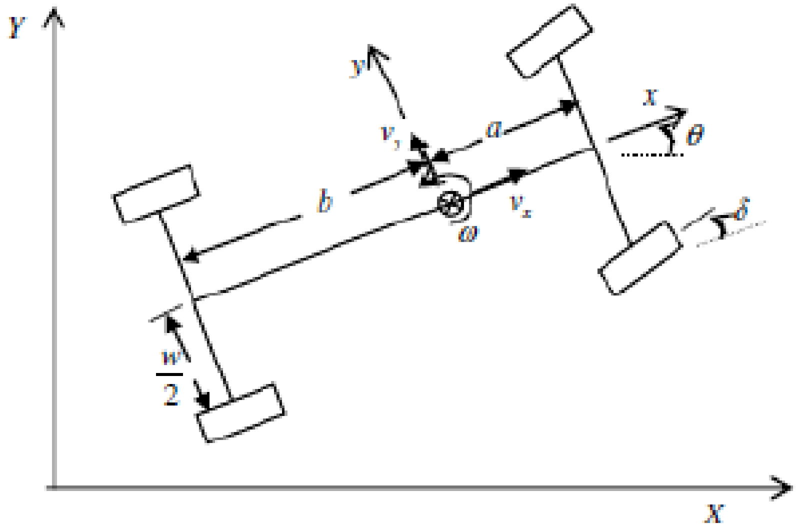
$$\vec{A}_p = \vec{A}_0 + \dot{\vec{\Omega}} \times \vec{R} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{R}\right)$$

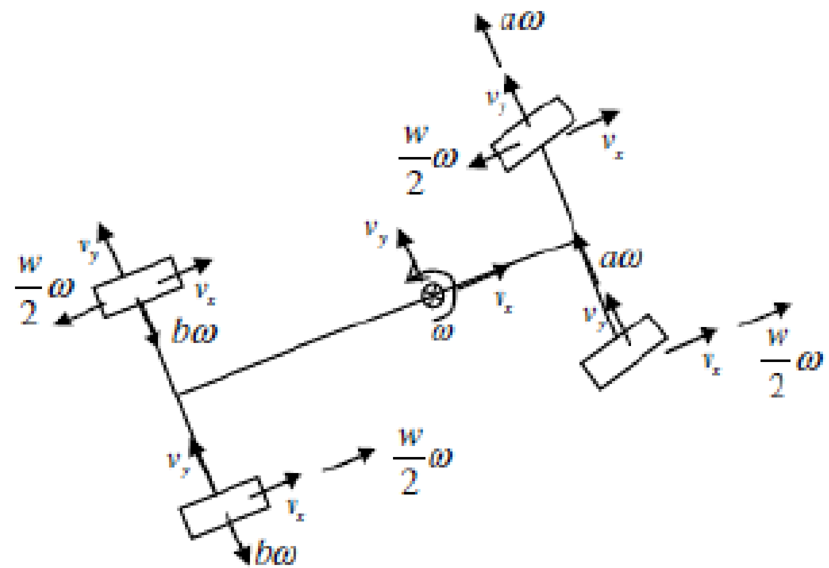




Example

- Velocity diagram at different wheels of a given configuration





- Under the condition of no side forces on the wheels

$$(v_y + a\omega)\cos\delta - (v_x + \frac{w}{2}\omega)\sin\delta = 0 \quad \text{at the front/right}$$

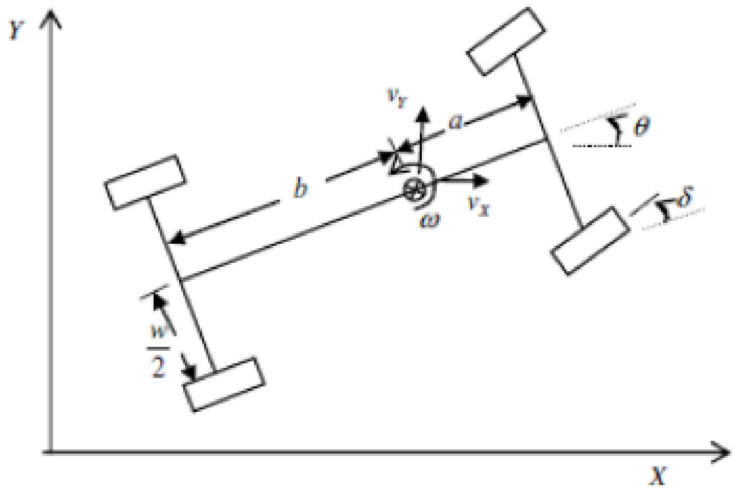
$$(v_y + a\omega)\cos\delta - (v_x - \frac{w}{2}\omega)\sin\delta = 0 \quad \text{at the front/left}$$

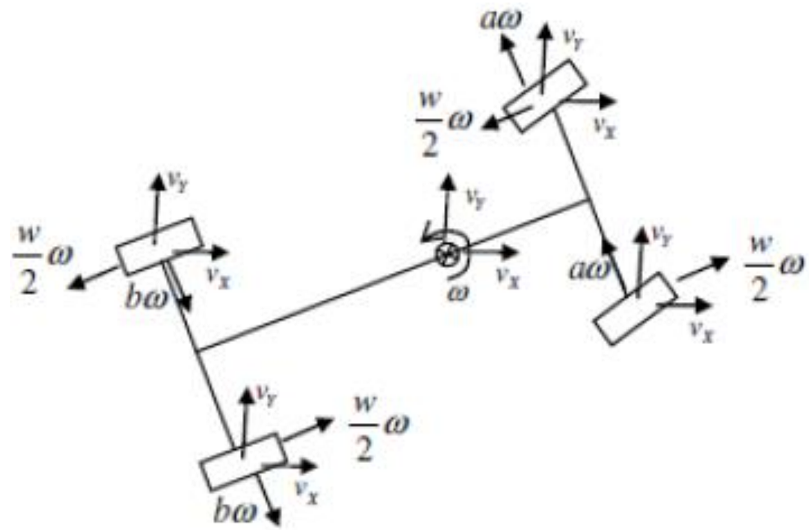
$$v_y - b\omega = 0 \quad \text{at the rear/right}$$

$$v_y - b\omega = 0 \quad \text{at the rear/left}$$

Example

- If the given velocities are prescribed w.r.t. the earth co-ordinate system





- Under the condition of no side forces on the wheels

$$v_Y \cos(\theta+\delta) + a\omega\cos\delta - v_X \sin(\theta+\delta) - \frac{w}{2}\omega\sin\delta = 0 \quad \text{at the front/right}$$

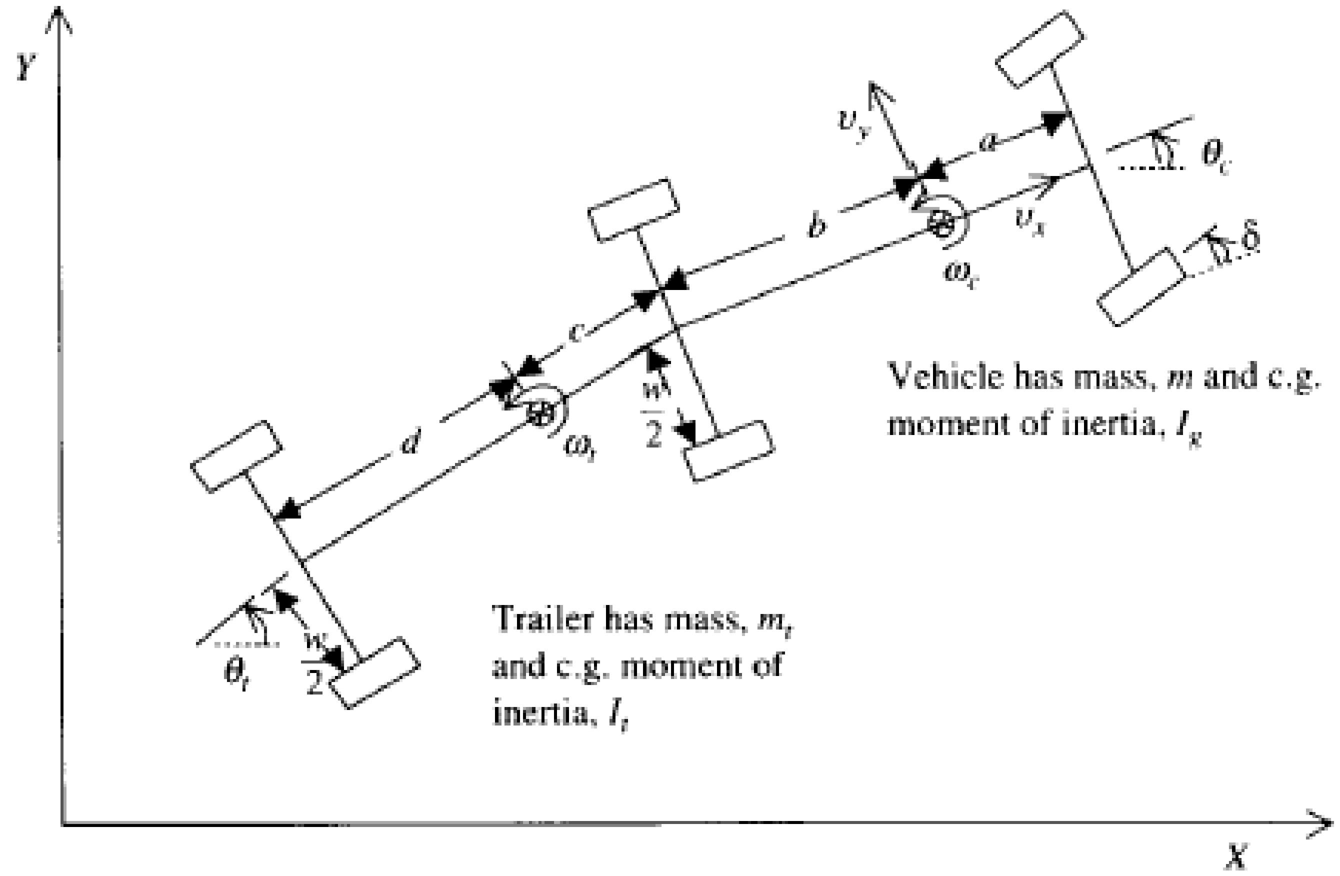
$$v_Y \cos(\theta+\delta) + a\omega\cos\delta - v_X \sin(\theta+\delta) + \frac{w}{2}\omega\sin\delta = 0 \quad \text{at the front/left}$$

$$v_Y \cos\theta - b\omega - v_X \sin\theta = 0 \quad \text{at the rear/right}$$

$$v_Y \cos\theta - b\omega - v_X \sin\theta = 0 \quad \text{at the rear/left}$$

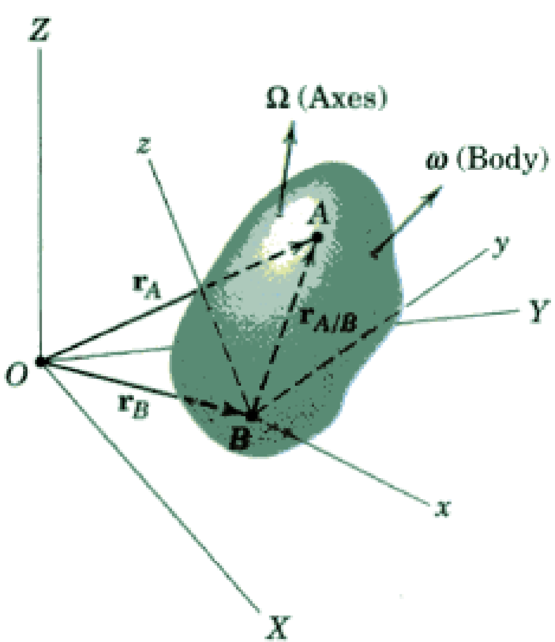
Practice Problem

1. Find out the velocity components at each wheel. Also mention the condition for no sideways force on each wheel for the following vehicle carrying a trailer. Assume required physical quantity.



Equation of Motion

- For a rigid body as shown in figure below, lets define the body fixed co-ordinate as shown below



$$\mathbf{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^{\dagger}$$

$$\mathbf{\Omega} = \begin{bmatrix} \Omega_x & \Omega_y & \Omega_z \end{bmatrix}^{\dagger}$$

Define: $\mathbf{p} \triangleq m\mathbf{v}$ (translational momentum)

Then: $\left(\frac{d\mathbf{p}}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{p}}{dt}\right)_{xyz} + \mathbf{\Omega}\times\mathbf{p}$

With $\mathbf{v}$ relative to rotating frame.

Newton's law: $\left(\frac{d\mathbf{p}}{dt}\right)_{XYZ} = \mathbf{F}$

Equation of motion for 6 DOF (3 trans. And 3 rot.)

$$\mathbf{F} = \left.\frac{d\mathbf{p}}{dt}\right|_{xyz} + \mathbf{\Omega}\times\mathbf{p} \qquad \mathbf{T} = \left.\frac{d\mathbf{h}}{dt}\right|_{xyz} + \mathbf{\Omega}\times\mathbf{h}$$

$$F_x = \dot{p}_x + \Omega_y p_z - \Omega_z p_y \qquad T_x = \dot{h}_x + \Omega_y h_z - \Omega_z h_y$$

$$F_y = \dot{p}_y + \Omega_z p_x - \Omega_x p_z \qquad T_y = \dot{h}_y + \Omega_z h_x - \Omega_x h_z$$

$$F_z = \dot{p}_z + \Omega_x p_y - \Omega_y p_x \qquad T_z = \dot{h}_z + \Omega_x h_y - \Omega_y h_x$$

Equation of Motion

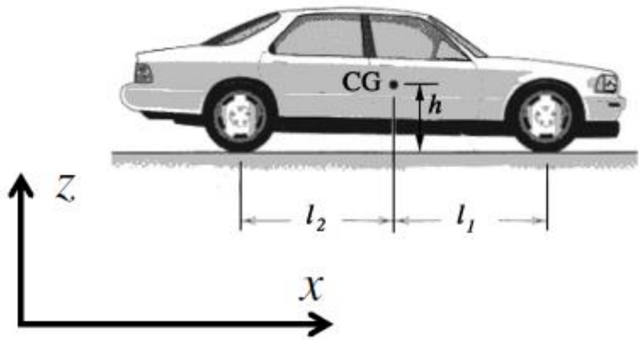
- State space for of the equations

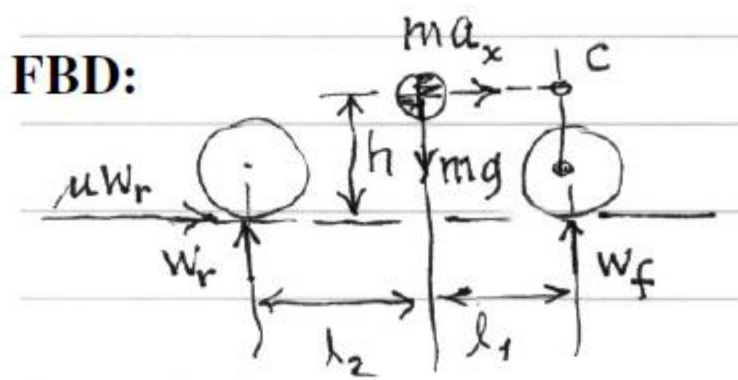
$$\dot{p}_x = F_x - \Omega_y p_z + \Omega_z p_y$$
$$\dot{p}_y = F_y - \Omega_z p_x + \Omega_x p_z$$
$$\dot{p}_z = F_z - \Omega_x p_y + \Omega_y p_x$$

$$\dot{h}_x = T_x - \Omega_y h_z + \Omega_z h_y$$
$$\dot{h}_y = T_y - \Omega_z h_x + \Omega_x h_z$$
$$\dot{h}_z = T_z - \Omega_x h_y + \Omega_y h_x$$

The terms which are not eligible can be neglected

- Example: Reduce the 6 DOF system to 1 DOF for the following problem





Solution

- Assuming no pitch, no roll, no yaw

$$\boxed{\dot{p}_x = F_x} - \cancel{\Omega_y} p_z + \cancel{\Omega_z} p_y$$

$$\dot{p}_y = 0 = F_y - \cancel{\Omega_z} p_x + \cancel{\Omega_x} p_z$$

$$\boxed{\dot{p}_z = 0 = F_z} - \cancel{\Omega_x} p_y + \cancel{\Omega_y} p_x$$

$$+\sum F_z = 0 = -W_f - W_r + W$$

$$\dot{h}_x = 0 = T_x - \cancel{\Omega_y} h_z + \cancel{\Omega_z} h_y$$

$$\boxed{\dot{h}_y = 0 = T_y - \cancel{\Omega_z} h_x + \cancel{\Omega_x} h_z}$$

$$\dot{h}_z = 0 = T_z - \cancel{\Omega_x} h_y + \Omega_y \cancel{h_x}$$

About CG:

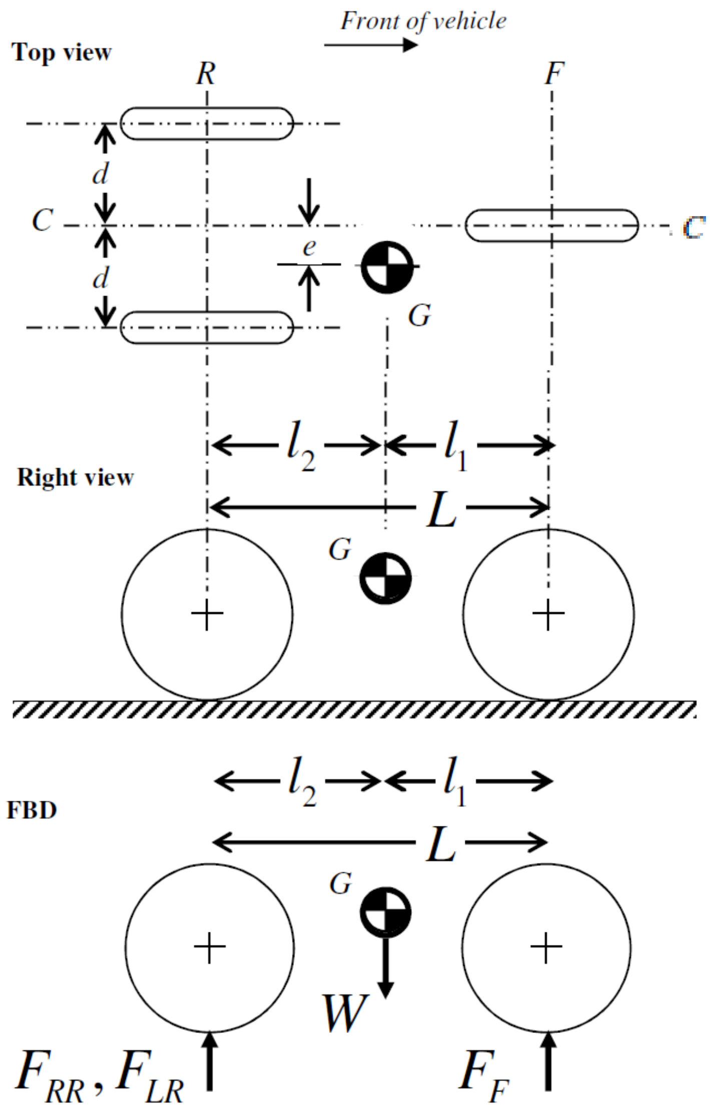
$$+\sum T_y = 0 = -W_r l_2 + W_f l_1 + \mu W_r h$$

$$\begin{bmatrix} 1 & 1 \\ -l_1 & l_2 - \mu h \end{bmatrix} \begin{bmatrix} W_f \\ W_r \end{bmatrix} = \begin{bmatrix} W \\ 0 \end{bmatrix} \Rightarrow W_f = \frac{l_2 - \mu h}{L - \mu h} W \text{ and } W_r = \frac{l_1}{L - \mu h} W$$

These two equations give two equations, two unknowns. Solve for the unknown forces, then apply to x-direction translational equation to find rate of change of forward velocity (acceleration).

Example

Find the weight distribution in a three-wheeled vehicle on level ground under static condition



Taking moment about R-R axis

$$+\text{ccw} \sum T_R = 0 = F_F L - W l_2 \quad \Rightarrow F_F = \frac{l_2}{L} W$$

Taking moment about C-C axis

$$+\text{ccw} \sum T_C = 0 = +We + F_{LR} d - F_{RR} d$$

$$\Rightarrow (F_{RR} - F_{LR}) = \frac{e}{d} W$$

Taking moment about F-F axis

$$+\text{ccw} \sum T_F = 0 = +W l_1 - (F_{RR} + F_{LR}) L$$

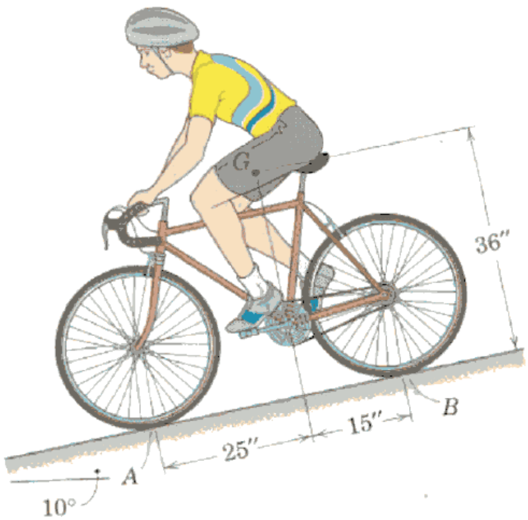
$$\Rightarrow (F_{RR} + F_{LR}) = \frac{l_1}{L} W$$

Solving for the rear axle forces

$$F_{RR} = \left(\frac{e}{d} + \frac{l_1}{L} \right) \frac{W}{2} \quad \text{and} \quad F_{LR} = \left(\frac{l_1}{L} - \frac{e}{d} \right) \frac{W}{2}$$

Example

Find the deceleration which would cause the tipping condition about the front wheel A?



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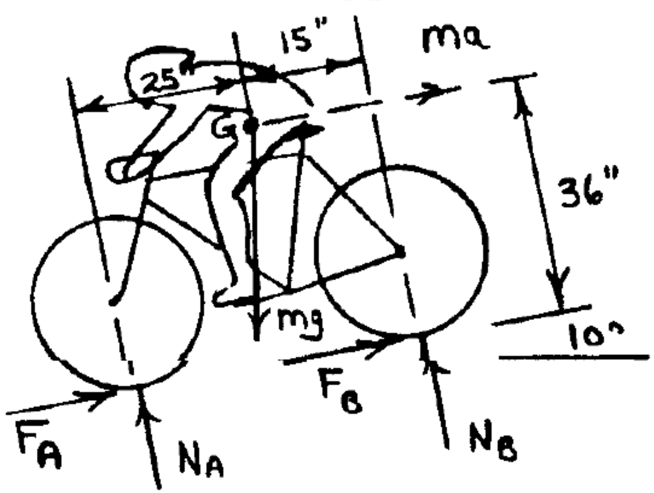


Solution: Tipping at the front wheel $N_B, F_B \longrightarrow 0$

$$\sum M_A = ma \times d$$

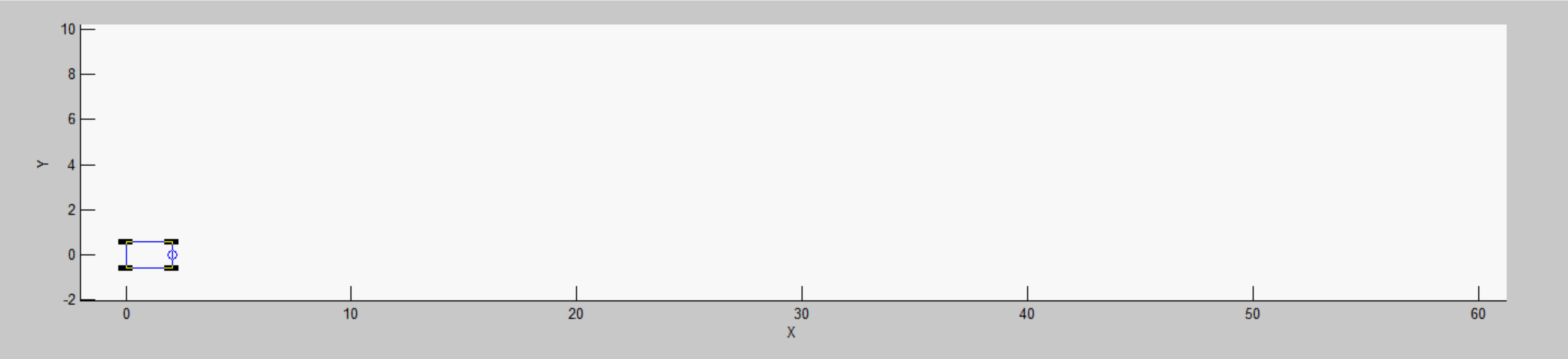
$$mg(25\cos 10^0 - 36\ \sin 10^0) = ma\,(36)$$

$$a = 0.510g$$

Deceleration of more than 0.510g will lead to Tipping condition

Practice Problem 2

Generate the animation for the path of a moving vehicle?

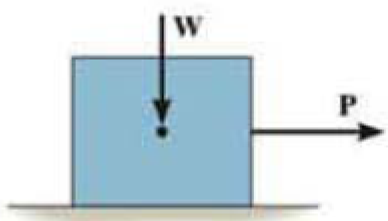


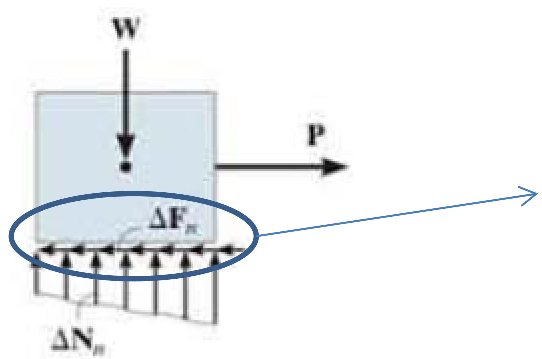
Importance of Sliding and Rolling Friction

Sliding and Rolling Friction

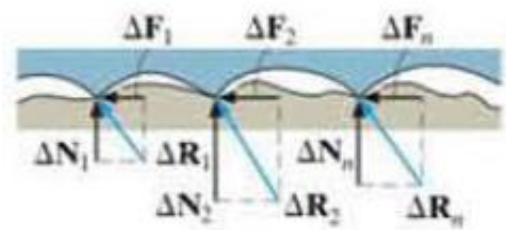
Sliding: Concept of static and kinetic (sliding friction)

1. Dry friction occurs between the contacting surfaces of bodies when there is no lubricating fluid.
2. Assumptions
   - Rough horizontal surface which is no n-rigid or deformable
   - Moving block having weight **W** is considered to be rigid
   - Block is pulled by a horizontal pulling load **P**





Microscopic observation!



Block weight =W

Pulling load = **P**

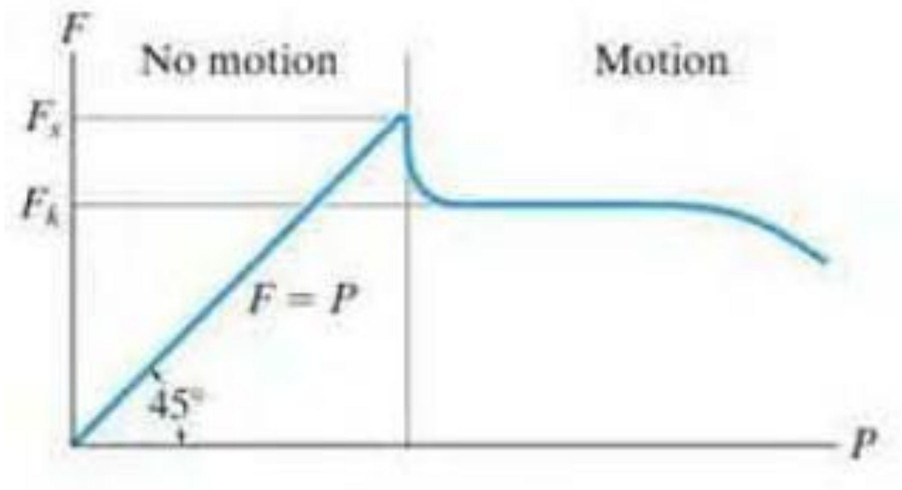
Normal force=ΔN_n

Frictional force=ΔF_n

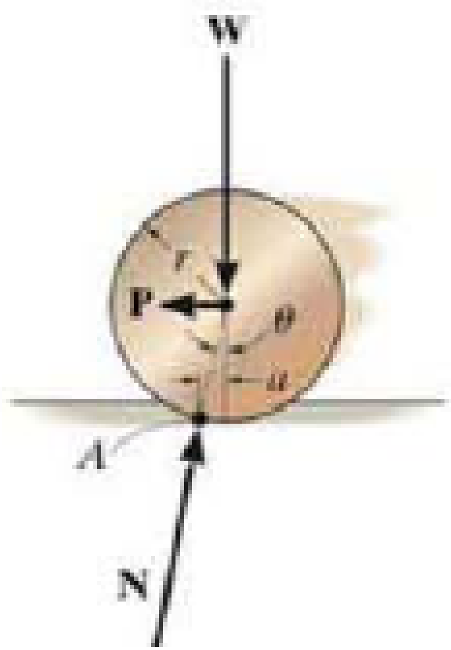
Normal force and Frictional force are non-uniform

Friction in a Nutshell

- F is a *static frictional force* if equilibrium is maintained.
- F is a *limiting static frictional force* F_s when it reaches a maximum value needed to maintain static equilibrium.
- F is a *kinetic frictional force* F_k when sliding occurs at the contacting surfaces.



Force Analysis of Rolling Body



- Resultant of distributed normal force

- To keep the cylinder in equilibrium, all the forces must be concurrent.

- Resultant force will pass through the center and making an angle of ɵ with vertical

- Taking a moment about A, we get

$$Wa = Pr\cos\theta$$

Assuming small ɵ, cos (ɵ) ≈ 1

$$Wa \approx Pr \quad \Longrightarrow \quad P \approx \frac{Wa}{r}$$

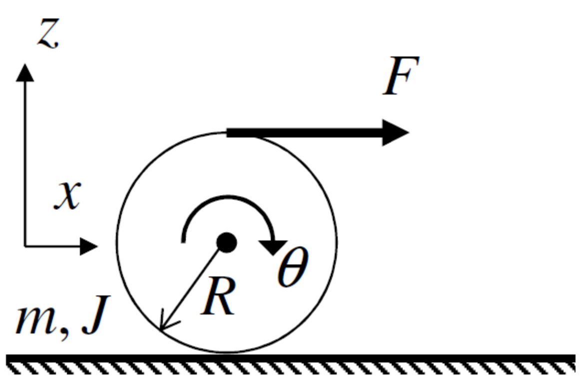
The distance *a* is termed as the coefficients of rolling resistance having the dimension of length.

Resisting torque: $T = Na$

Examples

Example: Wheel being pulled in pure roll

A homogeneous wheel of radius *R* and mass *m* is initially at rest on a rough horizontal surface. An external force *F* is applied at the top rim of the wheel as shown. Assuming the wheel rolls without sliding, find the magnitude and direction of the static friction force

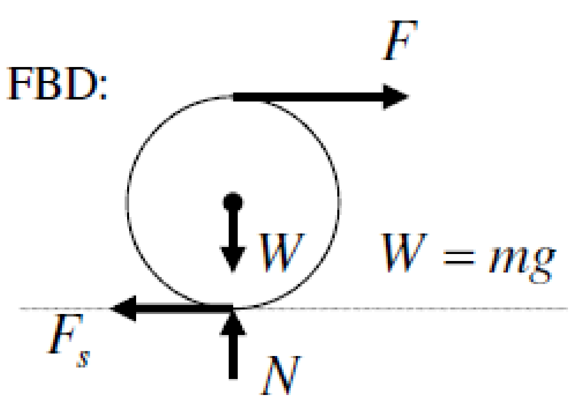


Solution: Assuming static force, we get

$$\dot{p}_x = m\dot{v}_x = m\ddot{x} = F - F_s$$

$$\dot{p}_z = m\dot{v}_z = N - W = 0$$

$$\dot{h}_y = J\dot{\omega}_y = J\ddot{\theta} = FR + F_s R$$



Since the cylinder rolls without sliding

$$x = R\theta \quad (\text{or, } \dot{v}_x = R\ddot{\theta})$$

$$mR\ddot{\theta} = F - F_s$$

$$J\ddot{\theta} = R(F + F_s)$$

$$\Rightarrow \frac{\frac{1}{2}mR^2}{mR}(F - F)_s = R(F + F_s)$$

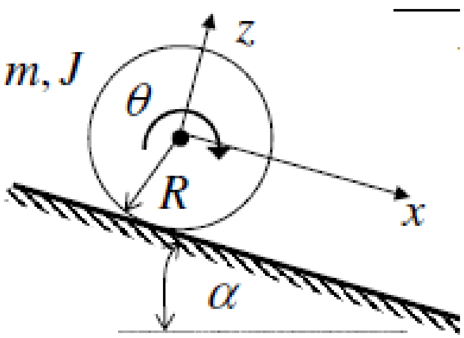
$$(F - F)_s = 2(F + F_s) \quad \Rightarrow \boxed{F_s = -\frac{1}{3}F}$$

Since, F is less than Fs, slip condition is valid. However, the sign should be opposite

Examples

Example: Wheel rolling down the incline with friction

A homogeneous wheel of radius *R* and mass *m* moves down an incline with inclination α. Find the angle α for which the wheel moves *without* sliding (or skidding).



Solution: When the wheel rolls without sliding (or slip),

$$F < \mu_k N = \mu_k mg\cos\alpha$$

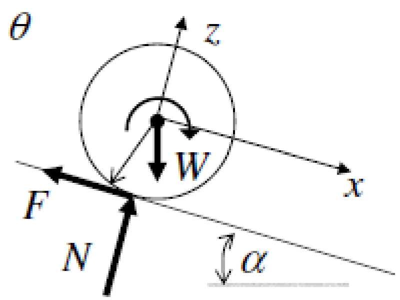
Equations of motion are:

$$\dot{p}_x = m\dot{v}_x = m\ddot{x} = mg\sin\alpha - F$$

$$\dot{p}_z = m\dot{v}_z = m\ddot{z} = N - W = N - mg\cos\alpha = 0$$

$$\dot{h}_y = J\dot{\omega}_y = J\ddot{\theta} = FR$$

FBD:



If the wheel rolls without slip, we also have $x = R\theta$

Taking $\boxed{J = \frac{1}{2}mR^2}$

$$\ddot{x} = R\ddot{\theta} \quad \Rightarrow J\frac{\ddot{x}}{R} = FR \quad \Rightarrow F = \frac{J}{R^2}\ddot{x} = \frac{1}{2}m\ddot{x}$$

$$\therefore m\ddot{x} = mg\sin\alpha - \frac{1}{2}m\ddot{x} \quad \Rightarrow \ddot{x} = \frac{2}{3}g\sin\alpha$$

$$\therefore F = \frac{1}{2}m\ddot{x} = \frac{1}{3}mg\sin\alpha < \mu_k N = \mu_k mg\cos\alpha$$

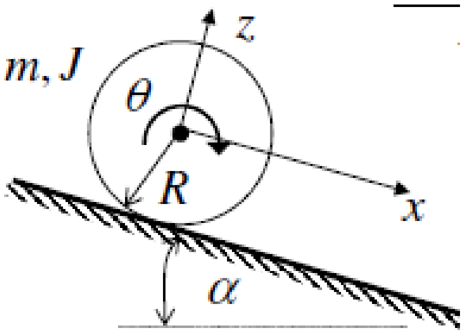
Defines the no-slip condition for α.

$$\frac{1}{3}\tan\alpha < \mu_k < \mu_s \quad \Leftarrow$$

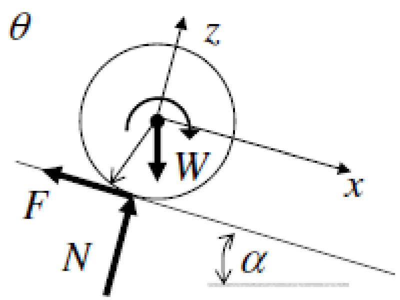
Practice Problem

Example: Wheel rolling down the incline with friction

A homogeneous wheel of radius *R* and mass *m* moves down an incline with inclination α. Find the angle α for which the wheel moves *without* sliding (or skidding).



FBD:



Solution: When the wheel rolls without sliding (or slip),

$$F < \mu_k N = \mu_k mg\cos\alpha$$

Equations of motion are:

$$\dot{p}_x = m\dot{v}_x = m\ddot{x} = mg\sin\alpha - F$$

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If the wheel rolls without slip, we also have $x = R\theta$

Taking $\boxed{J = \tfrac{1}{2}mR^2}$

$$\ddot{x} = R\ddot{\theta} \quad \Rightarrow J\frac{\ddot{x}}{R} = FR \quad \Rightarrow F = \frac{J}{R^2}\ddot{x} = \frac{1}{2}m\ddot{x}$$

$$\therefore m\ddot{x} = mg\sin\alpha - \frac{1}{2}m\ddot{x} \quad \Rightarrow \ddot{x} = \frac{2}{3}g\sin\alpha$$

$$\therefore F = \frac{1}{2}m\ddot{x} = \frac{1}{3}mg\sin\alpha < \mu_k N = \mu_k mg\cos\alpha$$

Defines the no-slip condition for α.

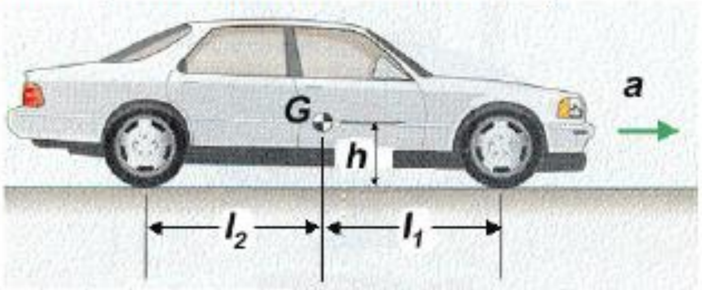
$$\frac{1}{3}\tan\alpha < \mu_k < \mu_s \quad \Leftarrow$$

Practice Problem

Example : Show that when the wheel rolls down the incline with sliding, x >$R\theta$

Example: finding reaction forces for two-axle vehicle in maximum acceleration

A two-axle vehicle with mass, m_v, at center G. is in maximum acceleration state. Find expressions for the total normal forces at the front and rear pairs of wheels, N_f and N_r , respectively. Assume the mass of the wheels is small compared with the total mass of the car, and that the coefficient of static friction between the road and the rear driving wheels is μ.



Example Solving for acceleration of powered mower for a given friction

The rear-wheel-drive lawn mower, when placed into gear while at rest, is observed to momentarily spin its rear tires as it accelerates. If the coefficients of friction between the rear tires and the ground are $\mu_s = 0.7$ and $\mu_k = 0.5$, determine the forward acceleration a of the mower. The mass of the mower and attached bag is 50 kg with center of mass at G. Assume that the operator does not push on the handle, so that $P = 0$.