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# Mass sensitivity of non-uniform microcantilever beams

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**Abstract**

Microelectromechanical systems (MEMS) based cantilever beams have been widely used in various sensing applications. Previous studies have aimed at increasing the sensitivity of biosensors by reducing the size of cantilever beams to nanoscale. However, the influence of non-uniform cantilever beams on mass sensitivity has rarely been investigated. In this paper, we discuss the mass sensitivity with respect to linear and nonlinear response of non-uniform cantilever beam with linear and quartic variation in width. To do the analysis, we use the nonlinear Euler-Bernouli beam equation with harmonic forcing. Subsequently, we derive the mode-shape corresponding to linear, undamped, free vibration case for different types of beams with a tip mass at the end. After applying the boundary conditions, we obtain the resonance frequencies corresponding to various magnitude of tip mass for different kinds of beams. To do the nonlinear analysis, we use the Galerkin approximation and the Method of Multiple Scales (MMS). Analysis of linear response indicates that the non-dimensional mass sensitivity increases considerably by changing the planar geometry of the beam as compared to uniform beam. At the same time sensitivity further increases when the non-uniform beam is actuated in higher modes. Similarly, the frequency shift of peak amplitude of nonlinear response for a given non-dimensional tip mass increases exponentially and decreases quadratically with tapering parameter, $\alpha$, for diverging and converging non-uniform beam with quartic variation in width respectively. For the converging beam, we also found an interesting monotonically decreasing and increasing behavior of mass sensitivity with tapering parameter $\alpha$ giving an extremum point at $\alpha = 0.5$. Overall analysis indicates a potential application of the non-uniform beams with quartic converging width for biomass sensor.

## 1 Introduction

Performance of most of the microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS) based resonant sensors and actuators are dependent on their linear and nonlinear dynamic characteristics near their resonance frequency [1, 2, 3, 4, 5, 6, 7, 8]. To detect the presence of bio-molecules, cantilever based resonant MEMS mass sensors are extensively used. To increase the sensitivity and resonance frequency, previous attempts have resorted to reducing

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the dimension of beams. At the same time, most of the resonators proposed till date employ either uniform cantilever beam or uniform clamped-clamped beam as the vibrating element. In this paper, we propose another class of beams with non-uniform section which has a tremendous ability to increase the resonance frequency.

There have been several studies associated with the linear and nonlinear studies of uniform beams [6, 7, 9, 10, 11] as well as non-uniform beams [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] for various application in vibrations and acoustics. Mabie and Roger [13] studied the vibrations of cantilever beams with linearly tapered thickness by obtaining the solution in terms of known Bessel functions and linearly tapered width using numerical integration. At the same time double tapered cantilever beams have been studied by Mabie *et al.* [14] and Lau [15] with tip mass, again by obtaining the mode shape in terms of known Bessel functions. Additionally, beam with special cases of tapering have been studied by obtaining the solution in terms of the Bessel Function by Auciello *et al.* [16], and hypergeometric functions by Wang [17]. William *et al.* [18] has studied the influence of symmetrically linear and parabolic tapered sections on the resonance frequency of axially loaded beams. Abrate [19] solved the linear governing equation for special type of non-uniform width which can be transformed into equivalent equation of uniform beam. Recently, Wang [20] has numerically obtained the frequency of a tapered cantilever beam of constant thickness and linearly tapered width. Anderson *et al.* [9] carried out an experimental and theoretical investigation of nonlinear planar responses of parametrically excited cantilever beams. The same authors also studied experimentally both transverse and planar nonlinear responses of cantilever beams [10]. These studies were however without any tip masses. To further improve the performance of cantilever based mass sensor, Kim *et al* [6] analyzed the nonlinear response of a uniform cantilever beam based mass sensor with added mass considering both geometrical as well as inertial nonlinearities. A close examination of the above studies reveal that many of them include the investigation of linear frequency of beam with tapered thickness [13] and the beam with both linearly tapered thickness and width [14, 15]. However, analytical solutions of beams with varying width alone are limited, perhaps, due to the difficulties in obtaining the exact form of the solution in this case. Moreover, nonlinear analysis corresponding to non-uniform beams with tip masses has not been investigated. Recently, we have developed an analytical model to analyze the influence of non-uniformity on the pull-in voltage and linear frequency at different DC bias of non-uniform cantilever beam without any tip mass [21].

In this paper, we deal with the linear and nonlinear frequency characteristics of non-uniform cantilever beam with tip mass as shown in Fig. 1. To perform nonlinear analysis, we use the Method of Multiple Scales (MMS) to obtain the approximate solution. To analyze the influence of tapering on the mass sensitivity corresponding to linear and nonlinear response of non-uniform beam with different types of tapering, we use generalized governing equation for non-uniform beam from our previous work Ref. [21] with tip mass condition. To validate the analytical model with tip mass, we compare results with the available models and the finite element model using ABAQUS. The analysis presented in the paper can prove to be a step forward towards employing non-uniform beams in order to achieve high sensitivity in MEMS sensors.

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# 2 Mathematical Theory

## 2.1 Governing Equation and Linear Frequency

To develop the analytical model, we use the nonlinear Euler beam equation governing the transverse vibration of a non-uniform cantilever beam with sinusoidal forcing given by [21].

$$\rho A(x)\ddot{w} - EI(x)(w^{''})^3 + w^{'}\rho A(x)\int_0^x (w'\ddot{w}' + (\dot{w}^{'})^2)dx$$
$$+w'(EI(x)w''w')'' + (EI(x)w'')'' + c\dot{w} = Q(t) \tag{1}$$

where, $c$ is the viscous damping coefficient, $Q(t) = F\sin(\Omega t)$ is the uniformly applied load as shown in Fig. 1, and over-dots and over-dash refer to differentiation with respect to time and space, respectively. Using the non-dimensional parameters

$$x^* = \frac{x}{L},\ w^* = \frac{w}{L},\ t^* = \frac{t}{(L^2\sqrt{\frac{\rho A_0}{EI_0}})}, f_1(x) = \frac{E\tilde{I}(x)}{EI_0}, f_2(x) = \frac{\rho\tilde{A}(x)}{\rho A_0}, \tag{2}$$

such that $EI(x) = EI_0 + E\tilde{I}(x) = (1 + f_1(x))EI_0,\ \rho A(x) = \rho A_0 + \rho\tilde{A}(x) = (1 + f_2(x))\rho A_0$, where, $I_0$ and $A_0$ are the area moment of inertia and cross sectional area at the fixed end of the cantilever beam respectively, we get the equivalent non-dimensional governing equation as (after dropping $^*$ for convenience)

$$(1 + f_2(x))\ddot{w} + ((1 + f_1(x))w^{''})^{''} - (1 + f_1(x))(w^{''})^3 + (1 + f_2(x))w^{'}\times$$
$$\int_0^x ((\dot{w}^{'})^2 + w^{'}\ddot{w}^{'})dx + c_1\dot{w} + w^{'}((1 + f_1(x))w^{''}w^{'})^{''} - \frac{L^3F}{EI_0}\sin(\Omega t) = 0. \tag{3}$$

The corresponding boundary conditions with added tip mass can be written as

$$w(0,t) = \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=1} = 0,\ \left.\frac{\partial}{\partial x}\Big(1 + f_1(x)\frac{\partial^2 w(x,t)}{\partial x^2}\Big)\right|_{x=1} = \mu\frac{\partial^2 w(1,t)}{\partial t^2} \tag{4}$$

where, $\mu = \frac{m}{\rho A_0 L}$, $m$ is the added tip mass and $c_1 = \frac{cL^2}{\sqrt{\rho A_0 EI_0}}$.

In order to find the linear undamped mode shape, we remove the nonlinear and damping terms of Eqn. (3) to obtain the linear governing equation for transverse vibrations, $w_{nu}$, of non-uniform beams. Since, the exact analytical solution of the resulting equation for a beam with tapered width is difficult to obtain, we introduce a function $\sigma(x)$ and convert equation for non-uniform equation to an equivalent equation for a uniform beam in terms of $w_u = \sigma w_{nu}$ as described in [19, 21]. By equating the coefficients of non-uniform and uniform equations, we find the expression of $\sigma(x)$, where, the higher derivative terms of $\sigma$ such as $\sigma^{''''}$, $\sigma^{'''}$, are either zero or negligible. We use the single mode approximation as $w_u(x,t) = \phi_u(x)\eta(t)$ and $w_{nu}(x,t) = \phi_{nu}(x)\eta(t)$, where $\phi_u$ and $\phi_{nu}$ are the mode shapes for uniform and non-uniform beams, respectively to obtain the resulting equation. The solution of this equation gives the modeshape which is solved with boundary condition from Eqn. (4) to obtain the modal frequencies and mode shapes for different types of beams. For a uniform beam, ($\sigma(x) = 1$) the modeshape after applying the boundary conditions is

$$\begin{aligned}\phi_u(x) = A_1\Bigg[\Lambda\Bigg(&(2\cos(\beta)\cosh(\beta)\beta\mu + \cos(\beta)\sinh(\beta) + \sin(\beta)\cosh(\beta))\sin(\beta x)\\ &+(2\cos(\beta)\sinh(\beta)\beta\mu + 1 + \cos(\beta)\cosh(\beta) + \sinh(\beta)\sin(\beta))\cosh(\beta x)\\ &-(2\cos(\beta)\cosh(\beta)\beta\mu + \cos(\beta)\sinh(\beta) + \sin(\beta))\\ &\times\cosh(\beta)\sinh(\beta x)\Bigg) + \cos(\beta x)\Bigg]\end{aligned} \tag{5}$$

where, $\Lambda = \Big[-\sin(\beta)\Big(2\beta\mu\cosh(\beta) + \sinh(\beta)\Big) + \cos(\beta)\cosh(\beta) + 1\Big]^{-1}$ and $\mu$ is the tip mass parameter. The modeshape is dependent on the tip mass $\mu$. The value of constant $A_1$ can be found by normalizing the modeshape using the relation $\int_0^1 \phi^2 dx = 1$.

Using similar methodology, we obtain the modeshape for non-uniform beams which effectively becomes $\phi_{nu} = \frac{\phi_u(x)}{\sigma(x)}$, where, $\sigma(x)$ is a function which captures the non-uniform effect in mode shape [21]. For a beam with linear variation in width, $\sigma(x) = \sqrt{1+\alpha x}$ and for quartic variation in width, $\sigma(x) = (1+\alpha x)^2$. The boundary conditions given in Eqn. (4) are applied on the modeshape $\phi_{nu}(x)$ to obtain the corresponding frequency equations. Subsequently, the frequency equation is solved numerically to obtain the different frequencies of the beams. Tables 1, 2 and 3 present the first three frequencies of uniform and non-uniform beams.

## 2.2 Modal Dynamic Equation

After obtaining the mode shapes corresponding to different tapered beams, we obtain the modal dynamic equation from Eqn. (3) to analyse the tapering effect on the nonlinear frequency response. Approximate the transverse deflection $w(x,t)$ by a single mode as $w(x,t) = \phi(x)\eta(t)$, where $\phi(x)$ is the linear undamped modeshape and $\eta(t)$ is time dependent variable. After substituting assumed solution, $w$, in Eqn. (3), and then applying the Galerkin method, we get the following form of generalized modal dynamic equation

$$\ddot{\eta} + S_1\eta + S_2\eta^3 + S_3\eta^2\ddot{\eta} + S_4\eta\dot{\eta}^2 + S_5\dot{\eta} + S_6\sin(\Omega t) = 0, \tag{6}$$

where, $S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_6,\ S_7,\ S_8$ are the constants given by

$$\begin{aligned}
S_0 &= \textstyle\int_0^1 (1+f_2(x))\phi(x)^2 dx\\
S_1 &= \tfrac{1}{S_0}\textstyle\int_0^1 \phi(x)\Big(f_1(x)''\phi(x)'' + 2f_1(x)'\phi(x)''' + (1+f_1(x))\phi(x)''''\Big)dx\\
S_2 &= \tfrac{1}{S_0}\textstyle\int_0^1 \Big(\phi(x)\phi(x)'\big(f_1(x)''\phi(x)''\phi(x)' + 2f_1(x)'\phi(x)'''\phi(x)'\\
&\quad + 2f_1(x)'(\phi(x)'')^2 + (1+f_1(x))\phi(x)''''\phi(x)'\\
&\quad + (3+3f_1(x))\phi(x)'''\phi(x)''\big) - (1+f_1(x)(\phi(x)'')^3\phi(x)\Big)dx\\
S_3 &= \tfrac{1}{S_0}\textstyle\int_0^1 (1+f_2(x))\phi(x)\phi(x)'\int_0^x(\phi(x)')^2 dx\\
S_4 &= \tfrac{1}{S_0}\textstyle\int_0^1 (1+f_2(x))\phi(x)\phi(x)'\int_0^x(\phi(x)')^2 dx\\
S_5 &= \tfrac{1}{S_0}\textstyle\int_0^1 \frac{\omega}{Q_f}\phi(x)^2 dx
\end{aligned}$$

$S_6 = -\frac{1}{S_0}\int_0^1 \frac{L^3 F}{EI_0}\phi(x)dx.$

The above coefficients changes with the tip mass quantity, $\mu$ and the non-uniformity of the cantilever beam which is captured by $f_1(x) = \frac{E\tilde{I}(x)}{EI_0}$ and $f_2(x) = \frac{\rho\tilde{A}(x)}{\rho A_0}$.

To obtain the solution of above equation, we apply the method of multiple scales [5]. Rescaling the nonlinear terms in nonlinear modal dynamic equation given by Eqn. (3) with small quantity $\epsilon$, the resulting dynamic equation with weak nonlinearity can be written as

$$\ddot{\eta} + S_1\eta + \epsilon S_2\eta^3 + +\epsilon S_3\eta^2\ddot{\eta} + \epsilon S_4\eta\dot{\eta}^2 + \epsilon S_5\dot{\eta} + \epsilon S_6\sin(\Omega t) = 0. \tag{7}$$

To obtain the modulation equation, we approximate the solution using multiple time scales, $T_0 = t$, $T_1 = \epsilon t$, as $\eta = \eta_1(T_0, T_1) + \epsilon^1\eta_2(T_0, T_1)$, where, $\ddot{\eta} = \Big(D_0^2 + 2\epsilon D_0 D_1\Big)\eta$; $\dot{\eta} = \Big(D_0 + \epsilon D_1\Big)\eta$ and $D_n^m = \frac{d^m}{dT_n^m}$. Substituting the approximate solution in the governing equation given by Eqn. (7) and comparing the coefficient of $\epsilon^0$ and $\epsilon^1$, we obtain linear homogenous equation in $\eta_1(T_0, T_1)$ and non-homogenous linear equation in $\eta_2(T_0, T_1)$ with frequency $\omega$. Assuming the solution of of homogenous equation as $\eta_1 = A(T_1)e^{(\iota\omega T_0)} + \overline{A}(T_1)e^{(-\iota\omega T_0)}$ and then substituting it in non-homogenous equation with $\Omega = \omega + \epsilon\sigma$, where, $\sigma$ is the detuning parameter, we obtain the modified form of non-homogenous equation. Eliminating the secular terms from the modified equation and we obtain the real and autonomous evolution equation by assuming $A(T_1) = \frac{1}{2}a(T_1)e^{i\omega\phi(T_1)}$. Taking $\theta = \sigma T_1 - \phi$ and $\dot{\theta} = \frac{d\theta}{dT_1}$, we get the following modulation equations

$$a\dot{\theta} = \sigma + \frac{3}{8}\omega S_3 a^3 - \frac{1}{8}\omega S_4 a^3 - \frac{3}{8\omega}S_2 a^3 - \frac{1}{2\omega}S_6\sin(\theta) \tag{8}$$

$$\dot{a} = -\frac{1}{2}S_5 a + \frac{1}{2\omega}S_6\cos(\theta). \tag{9}$$

The above equations can be solved to obtain stable and unstable nonlinear response using the continuation software, MATCONT, in MATLAB. Readers can refer to Appendix A for detailed mathematical discussion on obtaining the modulation equations.

# 3 Results and Discussions

In this section, we study the linear and nonlinear frequency response of uniform as well as non-uniform cantilever beam. We first discuss the variation in linear frequencies of different non-uniform beams with tip mass of different magnitudes. Subsequently, we study the effect of tapering and added mass on the nonlinear frequency response. It is important to note that the linear and nonlinear analysis carried out in this work is pertaining to applications in MEMS. However, since the analysis is presented in non-dimensional quantities, it can be applied to cantilever beams of different length scales for application even in the macro domain.

## 3.1 Linear Frequency Analysis

In this section, we discuss the effect of tip mass and the non-uniformity parameter on the linear frequency of different types of cantilever beams. Utilizing the mode shapes obtained in the

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previous section for various non-uniform cantilever beams, we determine the eigenvalues by solving the corresponding frequency equation. To compare the results with numerical values, we take beam of length, $L = 100$ $\mu$m, width at fixed end as, $b_0 = 40$ $\mu$m, and beam thickness as $h = 0.8$ $\mu$m. The width at free end is varied for different values of taper parameter. The Young's modulus, density and Poisson ratio of the beam are taken as $E = 160$ GPa, $\rho = 2300\text{kg/m}^3$, $\nu = 0.22$, respectively. Based on the finite element analysis with sufficient number of elements, we numerically compute non-dimensional resonance frequencies of first three modes of uniform and non-uniform beams.

For uniform beam, the mode shape is given by Eqn. (5) with $\sigma = 1$. The corresponding frequencies for first three modes for various non-dimensional tip mass, $\mu$, are mentioned in Table 1. It also compares results at various tip masses, with the results obtained by Kim *et al.* [20] for the first mode and with FEM results for higher mode frequencies with $\mu = 0.0$ and $\mu = 0.2$, respectively. The results are found to be in good agreement. To study the influence of non-uniform beams on frequency, we consider beams with linear and quartic variation in width. For decreasing cross section from the fixed end, i.e., converging beams, $\alpha$ is taken as negative. For increasing cross section from the fixed end, i.e., diverging beams, $\alpha$ is taken as positive. For $\alpha$ and tip mass $\mu$, we solve the frequency equation to find the frequencies and the corresponding modeshape.

Table 2 presents the first three non-dimensional frequencies for converging beams with linear variation. The computed frequencies are compared with the frequency obtained by Mabie and Rodger [13] and with the FEA results using ABAQUS for different $\alpha$'s corresponding to $\mu = 0.0$ and $\mu = 0.2$. In Tables 3 and 4, we present the frequencies for converging and diverging beams respectively, with $\alpha = 0.2$ and $\alpha = 0.4$ at different $\mu$. We observed that the results obtained using the proposed method agrees well with the FEA results as well as that of Mabie and Rodger [13] till certain $\alpha$. We noticed that the frequency deviates from that of FEA results as alpha goes beyond $\pm 0.4$. It is due to the approximation involved in neglecting higher order terms $\sigma''''$ and $\sigma'''$. For tip mass $\mu = 0.0$, the relative error as compared to the FEA results is 3.5% when $\alpha = -0.4$ and 22.5% when $\alpha = -0.6$. Similarly, for tip mass $\mu = 0.2$, the relative error as compared to the FEA results is 3.7% when $\alpha = -0.4$ and 24.6% when $\alpha = -0.6$. As a result the proposed method works well in case of linearly tapered beam only for smaller values of $\alpha$. The variation of fundamental frequency of linearly tapered beams with taper parameter at tip mass $\mu = 0.0,\ 0.05$ and 0.1 are also shown in Figures 2(a) and (b).

Similarly, Table 5 presents the first three frequencies of quartic beams with different $\alpha$ for $\mu = 0.0$ and $\mu = 0.2$. The results obtained by the proposed method is then compared with the FEA results obtained from ABAQUS for $\mu = 0.0$ and $\mu = 0.2$. Both the results are found to be in good agreement. Tables 6 and 7 lists the frequencies for both negative and positive $\alpha$ at different $\mu$ corresponding to $|\alpha| = 0.2$ and $|\alpha| = 0.4$. Figures 2(c) and (d) show the variation of frequency with $\alpha$ for $\mu = 0.0, 0.05, 0.1$. Like the beam with linear variation in width, we see that for a diverging beam, the frequency decreases with an increase in $\alpha$, while for a converging beam it increases with an increase in the magnitude of $\alpha$. For a diverging beam, the shift in the frequency for a cantilever beam without added mass ($\mu = 0.0$) and with added mass gradually reduces with increasing non-uniform parameter. However, for a converging beam, this shift increases sharply with the non-uniform parameter.

Figure 2(e) shows the variation of non-dimensional mass sensitivity, $S = \frac{\Delta f}{\Delta \mu}$, where, $\Delta f = \frac{\delta f_0}{f_0}$

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is the non-dimensional frequency shift and $\Delta\mu = \frac{\delta m}{\rho A_0 L}$ is the non-dimensional added mass. On comparing the variation of mass sensitivity, $S$, for converging and diverging tapered beam with linear and quartic variation in width in Fig. (2)(e), we found that $S$ increases for converging beam, but it decreases for diverging beam. For a quartic converging beam, we found that the non-dimensional mass sensitivity, $S$, drastically increases by an order of magnitude higher as compared to that for uniform beam when $\alpha$ varies from 0 to 0.6 as shown in Fig. (2)(e). Moreover, the variation in the frequency of higher modes as shown in Fig. (2)(f) shows that $S$ increases by around 5 times, if the non-uniform beam with quartic tapering at tapering ratio $\alpha = 0.6$ operates in its $5^{\text{th}}$ mode. This makes a converging beam with quartic variation, a better and suitable design for an effective mass sensor as compared to beam with uniform as well as linearly tapered width. Now, we investigate the influence of tapering parameter on the non-linear response under different tip masses.

### 3.2 Nonlinear Frequency Analysis

The evolution equations given by Eqns.(8) and (9) are obtained from the method of multiple scales (MMS) to do the non-linear analysis of different beams. The quality factor used in the analysis is 50 and as the damping term $S_5$ depends on the frequency of the beam, we have optimized the forcing in order to get nonlinear response. To validate the approximate solution obtained using MMS, we solve the governing equation given by Eqn.(6) using the Runge-Kutta method and compare its result with that of approximate solution. Figures 3(a)-(c) show the comparative results for uniform beam, converging beam with linear variation and converging beam with quartic variation in width. The results show that the modulation equation can be used to do further analysis of non-uniform beam. Figure 3(d) shows the nonlinear response and the shift in frequency corresponding to the peak due to the addition of mass. Figures 3(e)-(h) show the frequency shifts due to addition of non-dimensional mass, $\Delta\mu = 0.1$, for different taper ratios. For converging beam with linear variation in width as shown in Fig. 3(e), we observed that the frequency shift corresponding to the peak, $\Delta f$, of nonlinear response decreases with an increase in tapering. On the other hand, for a diverging beam with linear variation in width as shown in Fig. 3(f), $\Delta f$ with increase in tapering parameter. For a converging beam with quartic variation in width as shown in Fig. 3(g), $\Delta f$ first decreases and then starts increasing thereby attaining a minimum variation at $\alpha = 0.5$. The corresponding frequency shift, $\Delta f$, for diverging beam with linear and quartic variation in width increases linearly and exponential with tapering parameter $\alpha$. Based on the above analysis, we found that the shift in frequency corresponding to the peak of non-linear response due to addition of tip mass decreases in converging beams and increases in diverging beams. We have also found in the previous section that the shift in linear frequency due to change in tip mass increases with tapering in converging beams and decreases with tapering in diverging beams.

Finally, we state that we have analyzed the mass sensitivity based on the linear and nonlinear response of uniform and non-uniform cantilever beams with different types of tapering. We have noticed that both linear frequency as well as nonlinear frequency response show higher degree of mass sensitivity for non-uniform converging beams with quartic variation in width at $\alpha = -0.6$. The analysis presented in this work can form the basis for future design of non-uniform cantilever beams in the development of many sensors and actuators.

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# 4 Conclusions

In this paper, we have presented the mass sensitivity on linear frequency and nonlinear frequency response of uniform and non-uniform cantilever beams with different tapering. To do the analysis, we used the nonlinear governing equation for non-uniform beam including the nonlinear curvature effect. We then obtain the exact form of the mode shape for uniform and non-uniform beam with linear and quartic variation in width of the beam with and without tip mass. After obtaining the exact mode shape, we first obtain the linear frequencies of first three modes of beam with different types of tapering. Subsequently, using the first mode shape approximation, we derive the nonlinear modal dynamic equation. The nonlinear frequency response is obtained by solving nonlinear dynamical equation using the method of multiple scales for different types of beams. On analyzing the influence of tip mass $\mu$ as well as tapering parameter, $\alpha$, on the linear frequency and nonlinear frequency response, we found that the converging tapered beam with quartic variation in width can be used to increase the mass sensitivity remarkably.

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# A Appendix

## A.1 Method of Multiple Scales to obtain Modulation Equation

The re-scaled governing equation is given by

$$\ddot{\eta} + S_1\eta + \epsilon S_2\eta^3 + \epsilon S_3\eta^2\ddot{\eta} + \epsilon S_4\eta\dot{\eta}^2 + \epsilon S_5\dot{\eta} + \epsilon S_6\sin(\Omega t) = 0. \tag{A.1}$$

To obtain the modulation equation, we approximate the solution as

$$\eta = \eta_1(T_0, T_1) + \epsilon^1\eta_2(T_0, T_1) \tag{A.2}$$

where multiple time scales, $T_0 = t$, $T_1 = \epsilon t$ are used. Using the relations, $\ddot{\eta} = \Big(D_0^2 + 2\epsilon D_0 D_1\Big)\eta$; $\dot{\eta} = \Big(D_0 + \epsilon D_1\Big)\eta$, we substitute the approximate solution given by Eqn. (A.2) in the governing equation given by Eqn. (A.1). The coefficient of $\epsilon^0$ and $\epsilon^1$ are then compared to get the following two equations

$$\mathcal{O}(\epsilon^0): \qquad D_0^2\eta_1 + \omega^2\eta_1 = 0, \tag{A.3}$$

$$\begin{aligned}\mathcal{O}(\epsilon^1): D_0^2\eta_2 + \omega^2\eta_2 = -\Big(&2D_0D_1\eta_1 + S_2\eta^3 + S_3\eta^2(D_0^2\eta)\\ &+S_4\eta_1(D_0\eta_1)^2 + S_5D_0\eta + S_6\sin(\Omega t)\Big),\end{aligned} \tag{A.4}$$

where, $D_i^j = \frac{\partial^j}{\partial T_i^j}$, $\omega^2 = S_1$.

Assuming the solution of Eqn. (A.3) of the form

$$\eta_1 = A(T_1)e^{(\iota\omega T_0)} + \overline{A}(T_1)e^{(-\iota\omega T_0)} \tag{A.5}$$

and then substituting it in Eqn. (A.4) with $\Omega = \omega + \epsilon\sigma$, where, $\sigma$ is the detuning parameter, we get

$$\begin{aligned}D_0^2\eta_2 + \omega^2\eta_2 = &\Big(S_3A(T_1)^3\omega^2 + S_4A(T_1)^3\omega^2 - S_2A(T_1)^3\Big)\Big(e^{(i\omega T_0)}\Big)^3\\ &+\Big(-2i\dot{A}(T_1)\omega - 3S_3A(T_1)^2\overline{A}(T_1) + 3S_3A(T_1)^2\overline{A}(T_1)\omega^2\\ &-S_4A(T_1)^2\overline{A}(T_1)\omega^2 - iS_5A(T_1)\omega + \frac{1}{2}iS_6e^{i\sigma T_1}\Big)e^{i\omega T_0} + cc,\end{aligned} \tag{A.6}$$

where, $cc$ represents the complex conjugate part of the equation. To get the converged solution, we eliminate the secular terms from Eqn. (A.6) which gives complex modulation equation as

$$\begin{aligned}\Big(&-2i\dot{A}(T_1)\omega - 3S_3A(T_1)^2\overline{A}(T_1) + 3S_3A(T_1)^2\overline{A}(T_1)\omega^2\\ &-S_4A(T_1)^2\overline{A}(T_1)\omega^2 - iS_5A(T_1)\omega + \frac{1}{2}iS_6e^{i\sigma T_1}\Big) = 0.\end{aligned} \tag{A.7}$$

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To obtain the real and autonomous form of Eqn. (A.7), we substitute $A(T_1) = \frac{1}{2}a(T_1)e^{i\omega\phi(T_1)}$ in Eqn. (A.7) and then equate the real and imaginary parts, separately. Taking $\theta = \sigma T_1 - \phi$ and $\dot{\theta} = \frac{d\theta}{dT_1}$, we get the following modulation equations

$$a\dot{\theta} = \sigma + \frac{3}{8}\omega S_3 a^3 - \frac{1}{8}\omega S_4 a^3 - \frac{3}{8\omega}S_2 a^3 - \frac{1}{2\omega}S_6\sin(\theta) \tag{A.8}$$

$$\dot{a} = -\frac{1}{2}S_5 a + \frac{1}{2\omega}S_6\cos(\theta). \tag{A.9}$$

The above equations are the modulation equations which can be solved to obtain stable and unstable nonlinear response using the continuation software, MATCONT in MATLAB.

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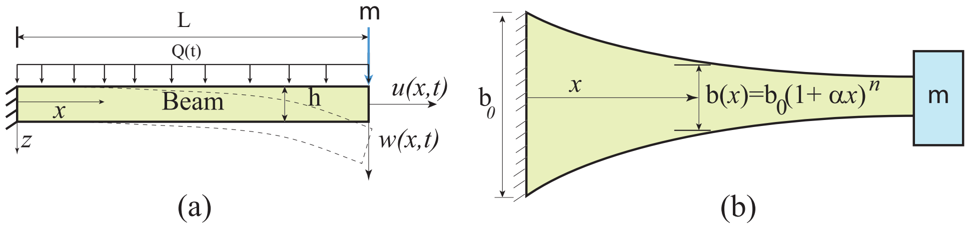


Figure 1: (a) Transverse vibration and axial stretching of a cantilever beam under uniformly distributed load. (b) The beam geometry used in our analysis. $\alpha = 0$ corresponds to a uniform beam, $\alpha > 0$ corresponds to diverging beam and $\alpha < 0$ corresponds to converging beams. Furthermore, $n = 1$ implies beam with linear variation in width and $n = 4$ implies beam with quartic variation in width.

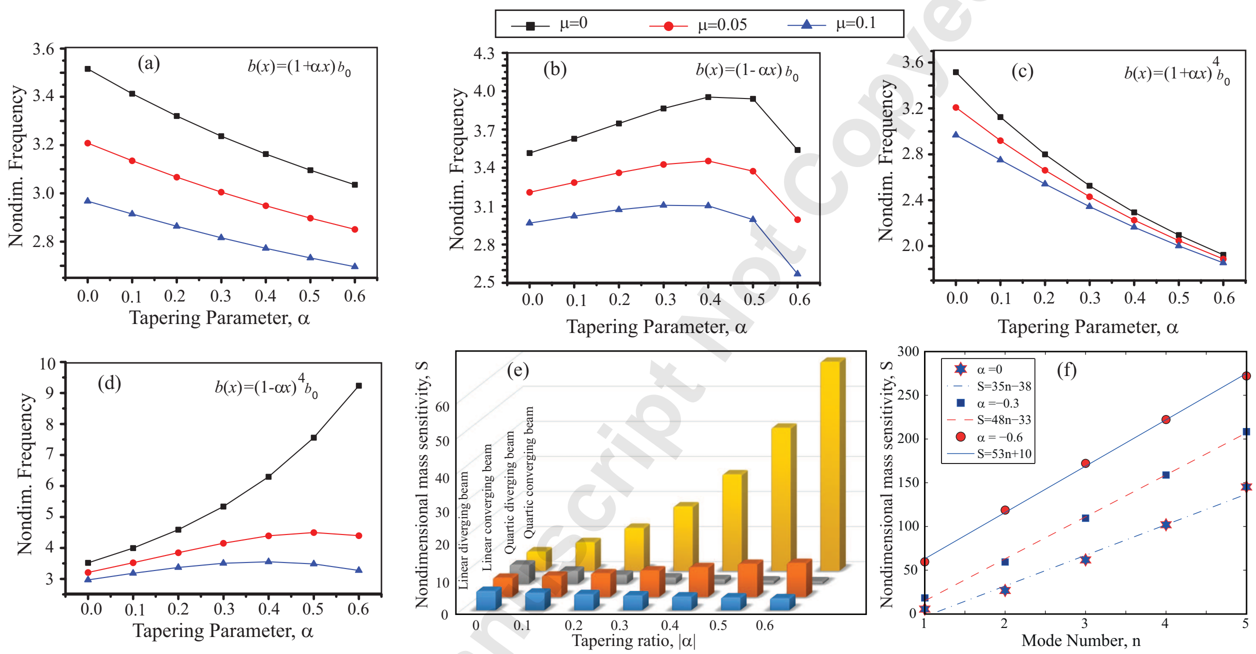


Figure 2: (a) Variation of non-dimensional mass sensitivity with tapering ratio for different types of tapering; (b) Variation of non-dimensional mass sensitivity versus mode number for uniform, $\mu = 0.0$, and non-uniform beam with quartic converging beams, $\alpha = -0.3$ and $\alpha = -0.6$.

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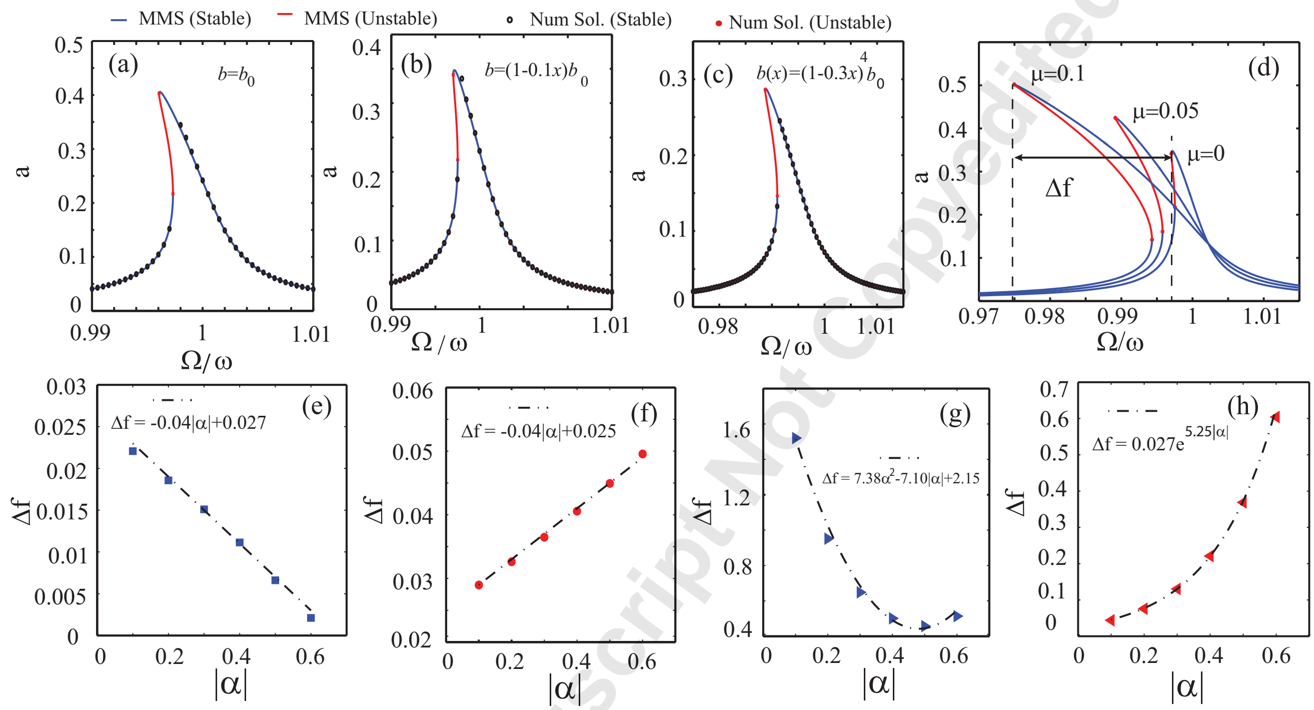


Figure 3: (a) Numerical and approximate solution for uniform beam, (b) converging beam with linear tapering, and (c) converging beam with quartic tapering. (d) Nonlinear response showing the shift in frequency due to added mass. (e) The frequency shift due to added mass at various $\alpha$ for converging beam with linear variation in width, (f) diverging beam with linear variation in width, (g) converging beam with quartic variation in width and (h) diverging beam with quartic variation in width.

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| $\mu$ | Mode | Present | Kim *et al.* [6] | ABAQUS |
|---|---|---|---|---|
| 0.0 | 1 | 3.5160 | 3.516 | 3.5292 |
| | 2 | 22.0345 | - | 22.0637 |
| | 3 | 61.6972 | - | 61.8717 |
| 0.1 | 1 | 2.9678 | 2.968 | - |
| | 2 | 19.3558 | - | - |
| | 3 | 55.5183 | - | - |
| 0.2 | 1 | 2.6127 | 2.613 | 2.6189 |
| | 2 | 18.2078 | - | 18.081 |
| | 3 | 53.5586 | - | 51.7742 |
| 0.3 | 1 | 2.3597 | 2.356 | - |
| | 2 | 17.5756 | - | - |
| | 3 | 52.6156 | - | - |
| 0.4 | 1 | 2.1679 | 2.168 | - |
| | 2 | 17.1763 | - | - |
| | 3 | 52.0632 | - | - |

Table 1: The nondimensional fundamental frequency of uniform beam with added mass.

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| $\alpha$ | Mode | Present Model | ABAQUS | Ref. [13] | | $\alpha$ | Mode | Present Model | ABAQUS | Ref. [13] |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 3.516 | 3.529 | 3.516 | | | 1 | 2.613 | 2.619 | 2.613 |
| 0 | 2 | 22.035 | 22.063 | 22.035 | | 0 | 2 | 18.208 | 18.081 | 18.208 |
| | 3 | 61.697 | 61.872 | 61.700 | | | 3 | 53.559 | 51.7749 | 53.550 |
| | 1 | 3.628 | 3.642 | - | | | 1 | 2.641 | 2.647 | - |
| -0.1 | 2 | 22.258 | 22.230 | - | | -0.1 | 2 | 18.213 | 18.094 | - |
| | 3 | 61.915 | 62.082 | - | | | 3 | 53.451 | 51.999 | - |
| | 1 | 3.747 | 3.771 | 3.717 | | | 1 | 2.662 | 2.677 | 2.720 |
| -0.2 | 2 | 22.523 | 22.516 | 22.415 | | -0.2 | 2 | 18.228 | 18.102 | 18.348 |
| | 3 | 62.180 | 62.280 | 62.060 | | | 3 | 53.354 | 52.146 | 53.580 |
| | 1 | 3.865 | 3.922 | - | | | 1 | 2.667 | 2.708 | - |
| -0.3 | 2 | 22.835 | 22.791 | - | | -0.3 | 2 | 18.260 | 18.102 | - |
| | 3 | 62.504 | 62.536 | - | | | 3 | 53.275 | 52.228 | - |
| | 1 | 3.954 | 4.098 | 3.892 | | | 1 | 2.633 | 2.740 | 2.810 |
| -0.4 | 2 | 23.207 | 23.105 | 22.743 | | -0.4 | 2 | 18.314 | 18.095 | 18.451 |
| | 3 | 62.910 | 62.811 | 62.390 | | | 3 | 53.227 | 52.255 | 53.580 |
| | 1 | 3.940 | 4.312 | - | | | 1 | 2.506 | 2.772 | - |
| -0.5 | 2 | 23.645 | 23.487 | - | | -0.5 | 2 | 18.402 | 18.077 | - |
| | 3 | 63.429 | 63.172 | - | | | 3 | 53.231 | 52.232 | - |
| | 1 | 3.5400 | 4.578 | 4.049 | | | 1 | 2.114 | 2.805 | 2.886 |
| -0.6 | 2 | 24.129 | 23.967 | 23.030 | | -0.6 | 2 | 18.542 | 18.046 | 18.530 |
| | 3 | 64.101 | 63.653 | 62.680 | | | 3 | 53.329 | 52.158 | 53.570 |

Table 2: The nondimensional frequency for beam with linear variation in width. (Left) When there is no tip mass i.e. $\mu = 0$. (Right) When tip mass $\mu = 0.2$.

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| $\mu$ | Mode | Present | Ref. [13] | $\mu$ | Mode | Present | Ref. [13] |
|---|---|---|---|---|---|---|---|
| 0.0 | 1 | 3.7475 | 3.7168 | 0.0 | 1 | 3.9542 | 3.8923 |
| | 2 | 22.5225 | 22.415 | | 2 | 23.2068 | 22.743 |
| | 3 | 62.1796 | 62.06 | | 3 | 62.9099 | 62.39 |
| 0.2 | 1 | 2.6621 | 2.7202 | 0.2 | 1 | 2.6329 | 2.8100 |
| | 2 | 18.2282 | 18.348 | | 2 | 18.3137 | 18.451 |
| | 3 | 53.3539 | 53.58 | | 3 | 53.2266 | 53.58 |
| 0.4 | 1 | 2.1729 | 2.2440 | 0.4 | 1 | 2.1060 | 2.3061 |
| | 2 | 17.2344 | 17.312 | | 2 | 17.3891 | 17.414 |
| | 3 | 52.0000 | 52.13 | | 3 | 52.0544 | 52.17 |
| 0.6 | 1 | 1.8809 | 1.9527 | 0.6 | 1 | 1.8052 | 2.0017 |
| | 2 | 16.7964 | 16.844 | | 2 | 17.0016 | 16.951 |
| | 3 | 51.4600 | 51.54 | | 3 | 51.6045 | 51.60 |
| 0.8 | 1 | 1.6816 | 1.7517 | 0.8 | 1 | 1.6047 | 1.7924 |
| | 2 | 16.5500 | 16.576 | | 2 | 16.7888 | 16.690 |
| | 3 | 51.1701 | 51.21 | | 3 | 51.3669 | 51.29 |

Table 3: The frequency of converging beam ($\alpha = -0.2$ and $\alpha = -0.4$) with linear variation in width.

| $\alpha = 0.2$ | $\mu$ | Mode | Freq. | $\alpha = 0.4$ | $\mu$ | Mode | Freq. |
|---|---|---|---|---|---|---|---|
| | 0.0 | 1 | 3.3205 | | 0.0 | 1 | 3.1626 |
| | | 2 | 21.6746 | | | 2 | 21.3998 |
| | | 3 | 61.3568 | | | 3 | 61.1044 |
| | 0.2 | 1 | 2.5523 | | 0.2 | 1 | 2.4963 |
| | | 2 | 18.2185 | | | 2 | 18.2446 |
| | | 3 | 18.2185 | | | 3 | 54.0184 |
| | 0.4 | 1 | 2.1471 | | 0.4 | 1 | 2.1248 |
| | | 2 | 17.1679 | | | 2 | 17.1866 |
| | | 3 | 52.1812 | | | 3 | 52.3258 |
| | 0.6 | 1 | 1.8881 | | 0.6 | 1 | 1.8809 |
| | | 2 | 16.6637 | | | 2 | 16.6605 |
| | | 3 | 51.4948 | | | 3 | 51.5796 |
| | 0.8 | 1 | 1.7046 | | 0.8 | 1 | 1.7051 |
| | | 2 | 16.3680 | | | 2 | 16.3461 |
| | | 3 | 51.1148 | | | 3 | 51.1605 |

Table 4: The frequency of diverging beam ($\alpha = 0.2$ and $\alpha = 0.4$) with linear variation in width.

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| $\alpha$ | Mode | Present Model | ABAQUS | | $\alpha$ | Mode | Present Model | ABAQUS |
|---|---|---|---|---|---|---|---|---|
| | 1 | 3.516 | 3.529 | | | 1 | 2.613 | 2.619 |
| 0 | 2 | 22.0345 | 22.064 | | 0 | 2 | 18.208 | 18.081 |
| | 3 | 61.697 | 61.872 | | | 3 | 53.559 | 51.774 |
| | 1 | 3.994 | 3.902 | | | 1 | 2.717 | 2.714 |
| -0.1 | 2 | 22.925 | 22.374 | | -0.1 | 2 | 18.200 | 18.094 |
| | 3 | 62.569 | 61.147 | | | 3 | 53.160 | 52.072 |
| | 1 | 4.587 | 4.466 | | | 1 | 2.780 | 2.768 |
| -0.2 | 2 | 23.960 | 23.330 | | -0.2 | 2 | 18.197 | 18.091 |
| | 3 | 63.620 | 62.006 | | | 3 | 52.898 | 52.204 |
| | 1 | 5.336 | 5.178 | | | 1 | 2.787 | 2.767 |
| -0.3 | 2 | 25.194 | 24.468 | | -0.3 | 2 | 18.242 | 18.112 |
| | 3 | 64.914 | 63.088 | | | 3 | 52.835 | 52.247 |
| | 1 | 6.298 | 6.240 | | | 1 | 2.726 | 2.699 |
| -0.4 | 2 | 26.709 | 26.504 | | -0.4 | 2 | 18.382 | 18.214 |
| | 3 | 66.554 | 66.081 | | | 3 | 53.022 | 52.394 |
| | 1 | 7.558 | 7.466 | | | 1 | 2.591 | 2.560 |
| -0.5 | 2 | 28.651 | 28.364 | | -0.5 | 2 | 18.655 | 18.439 |
| | 3 | 68.715 | 68.066 | | | 3 | 53.491 | 52.748 |
| | 1 | 9.235 | 9.096 | | | 1 | 2.380 | 2.349 |
| -0.6 | 2 | 31.282 | 30.901 | | -0.6 | 2 | 19.084 | 18.808 |
| | 3 | 71.728 | 70.902 | | | 3 | 54.271 | 53.338 |

Table 5: The nondimensional frequency for beam with quartic variation in width. (Left) When there is no tip mass i.e. $\mu = 0$. (Right) When tip mass $\mu = 0.2$.

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| $\alpha = -0.2$ | $\mu$ | Mode | Frequency | $\alpha = -0.4$ | $\mu$ | Mode | Frequency |
|---|---|---|---|---|---|---|---|
| | 0.0 | 1 | 4.5877 | | 0.0 | 1 | 6.2980 |
| | | 2 | 23.9601 | | | 2 | 26.7090 |
| | | 3 | 63.6203 | | | 3 | 66.5539 |
| | 0.2 | 1 | 2.7796 | | 0.2 | 1 | 2.7257 |
| | | 2 | 18.1968 | | | 2 | 18.3821 |
| | | 3 | 52.8975 | | | 3 | 53.0215 |
| | 0.4 | 1 | 2.1695 | | 0.4 | 1 | 2.018 |
| | | 2 | 17.3885 | | | 2 | 17.9269 |
| | | 3 | 51.9706 | | | 3 | 52.6062 |
| | 0.6 | 1 | 1.8391 | | 0.6 | 1 | 1.6746 |
| | | 2 | 17.0694 | | | 2 | 17.7655 |
| | | 3 | 51.6293 | | | 3 | 52.4631 |
| | 0.8 | 1 | 1.6245 | | 0.8 | 1 | 1.4623 |
| | | 2 | 16.8987 | | | 2 | 17.6830 |
| | | 3 | 51.4520 | | | 3 | 52.3906 |

Table 6: The frequency of converging beam ($\alpha = -0.2$ and $\alpha = -0.4$) with quartic variation in width.

| $\alpha = 0.2$ | $\mu$ | Mode | Frequency | $\alpha = 0.4$ | $\mu$ | Mode | Frequency |
|---|---|---|---|---|---|---|---|
| | 0.0 | 1 | 2.7993 | | 0.0 | 1 | 2.2943 |
| | | 2 | 20.5644 | | | 2 | 19.3823 |
| | | 3 | 60.3289 | | | 3 | 59.3020 |
| | 0.2 | 1 | 2.3414 | | 0.2 | 1 | 2.0539 |
| | | 2 | 18.1209 | | | 2 | 17.8439 |
| | | 3 | 54.4955 | | | 3 | 55.2865 |
| | 0.4 | 1 | 2.0517 | | 0.4 | 1 | 1.8754 |
| | | 2 | 17.0953 | | | 2 | 16.9797 |
| | | 3 | 52.6543 | | | 3 | 53.4432 |
| | 0.6 | 1 | 1.8478 | | 0.6 | 1 | 1.7363 |
| | | 2 | 16.5345 | | | 2 | 16.4284 |
| | | 3 | 51.7674 | | | 3 | 52.4000 |
| | 0.8 | 1 | 1.6945 | | 0.8 | 1 | 1.6241 |
| | | 2 | 16.1814 | | | 2 | 16.0465 |
| | | 3 | 51.2473 | | | 3 | 51.7317 |

Table 7: The frequency of diverging beam ($\alpha = 0.2$ and $\alpha = 0.4$) with quartic variation in width.