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Nonlinear coupling of transverse modes of a fixed-fixed microbeam under direct and parametric excitation

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Abstract Tuning of linear frequency and nonlinear frequency response of microelectromechanical systems (MEMS) are important in order to obtain high operating bandwidth. Linear frequency tuning can be achieved through various mechanisms such as heating, softening due to DC voltage, etc. Nonlinear frequency response is influenced by non-linear stiffness, quality factor, and forcing. In this paper, we present the influence of nonlinear coupling in tuning the nonlinear frequency response of two transverse modes of a fixed-fixed microbeam under the influence of direct and parametric forces near and below the coupling regions. To do the analysis, we use nonlinear equation governing the motion along in-plane and out-of-plane directions. For a given DC and AC forcing, we obtain static and dynamic equations using the Galerkin's method based on first mode approximation under the two different resonant conditions. First, we consider one-to-one internal resonance condition in which the linear frequencies of two transverse modes show coupling. Second, we consider the case in which the linear frequencies of two transverse modes are uncoupled. To obtain the non-linear frequency response under both the conditions, we solve the dynamic equation with the method of multiple scale (MMS). After validating the results obtained using MMS with the numerical simulation of modal equation, we discuss the influence of linear and nonlinear coupling on the frequency response of the in-plane and out-of-plane motion of fixed-fixed beam. We also analyzed the influence of quality factor on the frequency response of the beams near the coupling region. We found that the nonlinear response shows single curve near the coupling region with wider width for low value of quality factor, and it shows two different curves when the quality fac-

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tor is high. Consequently, we can effectively tune the quality factor and forcing to obtain different types of coupled response of two modes of a fixed-fixed microbeam.

1 Introduction

Microelectromechanical systems (MEMS) have been the subject of intense research in the design of sensitive sensors and actuators. The performance of MEMS based sensors and actuators are mainly dependent on their resonance frequencies. Hence, it is important to study the linear and nonlinear frequency tuning of such devices. While the linear frequency tuning can be achieved through various mechanisms such as hardening due to residual stresses, softening due to heating and DC voltage, and their combined effect, etc., the nonlinear frequency response can be tuned due to nonlinear stiffness, quality factor and forcing [1–5]. In this paper, we discuss about the tuning of nonlinear frequency response of a fixed-fixed microbeam under the direct and parametric excitation by controlling the linear and nonlinear stiffness through the coupling of two transverse modes and their quality factors at different excitation force.

Many researchers have analyzed the linear and nonlinear frequency response of in-plane or out-of-plane motion of a fixed-fixed or cantilever beam using one degree of freedom model. Younis and Nayfey [6], and Nayfeh *et al.* [7] studied the linear and nonlinear response of out-of-plane motion of a fixed-fixed beam subjected to direct forcing. Dumitru *et al.* [8] analyzed the nonlinear behavior of out-of-plane motion of a cantilever beam subjected to the fringing and direct forces. Linzon *et al.* [10] studied the parametric response of a cantilever beam in out-of-plane direction under the influence of fringing forces from two symmetrically placed side electrodes. Gutschmidt and Gottlieb [11, 12] studied the in-plane motion of an array of fixed-fixed beams under the parametric excitation. Lifshitz *et al.* [13] analyzed the parametric response of an array of fixed-fixed beams along in-plane direction and compare the results with experiments conducted by Buks and Roukes [14]. Kambali and Pandey [15] studied nonlinear response of out-of-plane motion of a fixed-fixed beam under the combined effect of direct and fringing forces. While the direct force leads to the nonlinear Duffing response, the fringing force induces parametric response. The combined effect of direct and fringing force results in the nonlinear Duffing response with enhanced amplitude as well as frequency width. To analyze the influence of the coupling of two or more modes, Nayfeh *et al.* [16] analyzed the nonlinear response of longitudinal and transverse motion of a taut string subjected to end excitation. Daqaq *et al.* [17] studied the linear and nonlinear coupled behavior of torsional micromirror when its torsional and transverse modes show 2:1 internal resonance condition. Isacsson *et al.* [18] presented numerical and analytical study of the linear and nonlinear coupled behavior of longitudinal and transverse motion of an array of carbon nanotube with fixed-free condition under parametric excitation. Samanta *et al.* [19] studied nonlinear coupling between various transverse modes of a MoS_2 nanomechanical beam under 1:1, 1:2 and 1:3 internal resonance conditions. Recently, Ramini *et al.* [9] presented experimental studies of primary and parametric resonances of a MEMS arch resonator. Wie *et al.* [20] investigated the weak and strong coupling in a periodically driven Duffing resonator elastically coupled to a van der Pol oscillator under 1:1 internal resonance condition. Matheny *et al.* [21] studied intra- and intermodal nonlinear coupling of a doubly clamped piezoelectric beam. Westra *et al.* [22] presented theoretical and experimental studies of nonlinear intermodal coupling between the flexural vibration modes of a single clamped-clamped beam. Conley *et al.* [23] analyzed the nonlinear dynamics of fixed-fixed nanowire and found the transition from a planer motion to whirling motion on increasing the excitation amplitude. Mahboob *et al.* [24] analyzed the nonlinear coupling of nanomechanical resonators by the coupled Vander Pol-Duffing equations. In this paper, we model and analyze the nonlinear coupling of two transverse modes of a fixed-fixed beam under the condition of 1:1 internal resonance.

To do nonlinear coupled analysis of in-plane and out-of-plane modes near and away from the coupling region, we consider the dimensions and properties of a fixed-fixed beam separated by two side electrodes and a bottom electrode as described by Kambali *et al.* [2]. The electrostatic force along the out-of-plane direction is based on direct forcing between the bottom electrode and the beam, and the parametric forcing between the beam and side electrodes. The force along in-plane direction is pure parametric forcing between the beam and the symmetrically placed side electrodes. Under the electrostatic forcing, we apply Galerkin's method to governing equations along two directions and obtain the reduced order form of corresponding static and dynamics equations. To obtain the condition of coupling, we take appropriate value of DC voltage such that the linear frequencies of in-plane mode, ω_1, and out-of-plane mode, ω_1, show coupling. To obtain the nonlinear coupled response, we solve the modal dynamic equation using the method of multiple scales under the condition of $\omega_1 \approx \omega_2$. After validating the multiple scale solution with numerical results obtained by solving modal dynamic equations, we analyze the influence of quality factor on nonlinear frequency response near the coupling region.

2 Governing equations

To present the partial differential equations governing the in-plane and out-of-plane motions of a fixed-fixed microbeam, we consider a beam of length L, width B and thickness H, which is separated from the two side electrodes E_1 and E_2 by gaps of g_0 and g_1, respectively, and the bottom electrode E_g by a gap of d as shown in Fig. 1 (a) and (b). Taking the deflection of the beam along in-plane and out-of-plane as $y(x,t)$ and $z(x,t)$, respectively, as shown in Fig. 1(a), the governing equation of motion along in-plane and out-of-plane directions considering damping, residual tension and mid-plane stretching [1] can be written as:

$$EI_{\bar{z}}\bar{y}^{''''} + \rho A\ddot{\bar{y}} + C_1\dot{\bar{y}} - \left[N_0 + \frac{EA}{2L}\int_0^L \left(\bar{z}^{'2} + \bar{y}^{'2}\right)d\bar{x}\right]\bar{y}_n^{''} = Q_{\bar{y}}(\bar{y},\bar{z},\bar{t}) \qquad (2.1)$$

$$EI_{\bar{y}}\bar{z}^{''''} + \rho A\ddot{\bar{z}} + C_3\dot{\bar{z}} - \left[N_0 + \frac{EA}{2L}\int_0^L \left(\bar{z}^{'2} + \bar{y}^{'2}\right)d\bar{x}\right]\bar{z}_n^{''} = Q_{\bar{z}}(\bar{y},\bar{z},\bar{t}) \qquad (2.2)$$

where, the subscripts prime and dot represent differentiation with respect to x and t, respectively, N_0 is the initial tension induced in the beam by fabrication processes and heating [2], E is the Young's modulus of the beam, EI is the bending

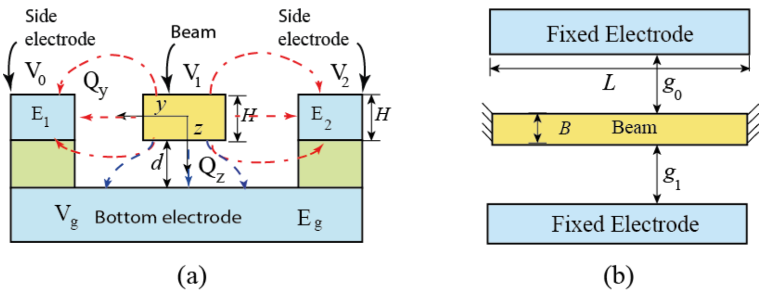

Fig. 1 (a) A fixed-fixed beam of width B, thickness H separated from the side electrodes, E_1 and E_2, by g_0 , g_1 and the ground electrode E_g by distance d are subjected to direct force, Q_z and fringing field force, Q_y; (b) Top view of a fixed-fixed beam of length L. separated from the side electrodes, E_1 and E_2, by g_0 , g_1, respectively.

rigidity, $I_z = HB^3/12$, $I_y = BH^3/12$ are area moment of inertia about z and y-axes, and ρ is the material density. The boundary conditions for the fixed-fixed beam are taken as

$$\bar{y}(0,t) = \bar{y}(L,t) = 0, \bar{z}(0,t) = \bar{z}(L,t) = 0, \bar{y}^{'}(0,t) = \bar{y}^{'}(L,t) = 0$$
$$\bar{z}^{'}(0,t) = \bar{z}^{'}(L,t) = 0. \qquad (2.3)$$

The forcing Q_y and Q_z are the effective electrostatic forces per unit length along y and z directions for the beam under the direct and fringing field effect as shown in Fig. 1 (a). The expressions for the forcing are given by [2]

$$Q_{\bar{y}}(\bar{y},\bar{z},\bar{t}) = \frac{1}{2}k_1\epsilon_0 H\left[\frac{(V_{10}+v(t))^2}{(g_0-\bar{y})^2} - \frac{(V_{12}+v(t))^2}{(g_1+\bar{y})^2}\right] \qquad (2.4)$$

$$Q_{\bar{z}}(\bar{y},\bar{z},\bar{t}) = \frac{1}{2}\frac{{V_{1g}}^2\epsilon_0}{B^2\,(d-\bar{z})^2}\left[4.32\,B^3 + 0.0182\,B\big(d-\bar{z}\big)^2 - k_2 0.00068\,(d-\bar{z})^3\right]$$
$$-\frac{1}{2}\frac{\epsilon_0 H}{g_0 B^2}\big(k_3 0.156\bar{z} + 0.0049B\big)\left[(V_{10}+v(t))^2 + (V_{12}+v(t))^2\right] \qquad (2.5)$$

where, $\epsilon_0 = 8.85\times10^{-12}$ F/m is the vacuum permittivity. Here, k_1 contributes for the net effect of fringing and direct fields in y-direction, k_2 and k_3 represent the strength of the fringing field effects from the bottom electrode and two side electrodes in the z-direction deflection. Further details regarding the extraction of electrostatic force parameters can be found in [25]. $V_{ij} = V_i - V_j$ is the voltage difference between the beam and electrodes and $v(t) = V_{ac}\cos(\Omega t)$.

2.1 Non-dimensionalization

To obtain the non-dimensional form of the governing equations, we define the ratio of the beam and side electrode gaps as $r_0 = (g_0/g_0)$, $r_1 = (g_1/g_0)$ and use the variables $x = \bar{x}/L$, $y = \bar{y}/g_0$, $z = \bar{z}/d$, $t = \bar{t}/T$, where, $T = \sqrt{\rho A L^4/EI_z}$. Finally, the non-dimensional nonlinear dynamic equations along the in-plane and out-of-plane directions for fixed-fixed beam under direct and parametric excitation can be written as:

$$y^{''''}+\ddot{y}+c_1\dot{y}-[N+\alpha_1\varGamma(y,y)+\alpha_2\varGamma(z,z)]y^{''}=\beta_s\Big[\frac{(V_{10}+v(t))^2}{(1-y)^2}-\frac{(V_{12}+v(t))^2}{(r_1+y)^2}\Big] \quad (2.6)$$

$$z^{''''}+\alpha_3\ddot{z}+c_3\dot{z}-\alpha_3[N+\alpha_1\varGamma(y,y)+\alpha_2\varGamma(z,z)]z^{''}=(\beta_g+\beta_{2g}(1-z)^2-\beta_{3g}(1-z)^3)\frac{(V_{1g}+v(t))^2}{(1-z)^2}-(\alpha_g+\alpha_{2g}z)[(V_{10}+v(t))^2+(V_{12}+v(t))^2]. \quad (2.7)$$

The corresponding non-dimensional form of the boundary conditions can be written as

$$y(0,t)=y(1,t)=0,\ z(0,t)=z(1,t)=0,\ y^{'}(0,t)=y^{'}(1,t)=0,\ z^{'}(0,t)=z^{'}(1,t)=0. \quad (2.8)$$

The various non-dimensionalised parameters as mentioned in above equations are defined as

$$\varGamma(p(x,t),q(x,t))=\int_0^1\frac{\partial p}{\partial x}\frac{\partial q}{\partial x}\mathrm{d}x, N=\frac{N_0L^2}{EI_z},\alpha_1=\frac{6g_0^2}{B^2},\alpha_2=\frac{6d^2}{B^2},\alpha_3=\left(\frac{I_z}{I_y}\right),$$
$$\beta_s=\frac{6k_1\sigma_1}{B^3g_0^3},\beta_g=\frac{25.92\sigma_1}{H^3d^3},\beta_{2g}=\frac{0.1092\sigma_1}{B^2H^3d},\beta_{3g}=\frac{k_2 4.08\times10^{-3}\sigma_1}{H^3B^3},\ \sigma_1=\frac{\epsilon_0L^4}{E},$$
$$\alpha_g=\frac{0.0294\sigma_1}{g_0B^2H^2d},\ \alpha_{2g}=\frac{k_3 9.36\times10^{-1}\sigma_1}{g_0B^3H^2},\ c_1=\frac{C_1L^4}{EIz\,T},\ c_3=\frac{C_3L^4}{EIy\,T\alpha_3}. \quad (2.9)$$

Since, the beam deflection is based on its static component due to DC voltage and dynamic component due to AC voltage, the deflection along y and z directions can be written as [1]

$$y(x,t)=u_s(x)+u(x,t),\ z(x,t)=w_s(x)+w(x,t). \quad (2.10)$$

where, $u_s(x)$ and $w_s(x)$ are static deflections. By substituting Eqn. (2.10) in Eqns. (2.6) and (2.7), and subsequently setting the time derivatives and dynamic forcing terms equal to zero, we get the static equations as:

$$u_s^{''''}-[N+\alpha_1\varGamma(u_s,u_s)+\alpha_2\varGamma(w_s,w_s)]\,u_s^{''}=\beta_s\Big[\frac{(V_{10})^2}{(1-u_s)^2}-\frac{(V_{12})^2}{(r_1+u_s)^2}\Big] \quad (2.11)$$

$$w_s^{''''}-\alpha_3\,[N+\alpha_1\varGamma(u_s,u_s)+\alpha_2\varGamma(w_s,w_s)]\,w_s^{''}=(\beta_g+\beta_{2g}(1-w_s)^2-\beta_{3g}(1-w_s)^3)\frac{(V_{1g})^2}{(1-w_s)^2}-\alpha_g[(V_{10})^2+(V_{12})^2]-\alpha_{2g}[(V_{10})^2w_s+(V_{12})^2w_s]. \quad (2.12)$$

Similarly, substituting Eqn. (2.10) in Eqns. (2.6) and (2.7), expanding the forcing terms about $u_n=0$ and $w_n=0$ upto the first order and subtracting the contribution of the nonlinear static terms given by Eqns. (2.11) and (2.12), we get the nonlinear dynamic equations for beam as,

$$u'''' + u_{tt} + c_1 u_t - [\alpha_1 \Gamma(u,u) + 2\alpha_1 \Gamma(u_s,u) + \alpha_2 \Gamma(w,w) + 2\alpha_2 \Gamma(w_s,w)]u_s'' \\ -[N + \alpha_1 \Gamma(u_s,u_s) + \alpha_1 \Gamma(u,u) + 2\alpha_1 \Gamma(u_s,u) + \alpha_2 \Gamma(w_s,w_s) + \alpha_2 \Gamma(w,w) \\ +2\alpha_2 \Gamma(w_s,w)]u'' = 2\beta_s \Big[\frac{(V_{10})^2 u}{(1-u_s)^3} + \frac{(V_{12})^2 u}{(r_1+u_s)^3}\Big] + \beta_s \frac{2V_{10}v(t)+v(t)^2}{(1-u_s)^2} \\ \Big[1 + \frac{2u}{(1-u_s)}\Big] - \beta_s \frac{2V_{12}v(t)+v(t)^2}{(r_1+u_s)^2}\Big[1 - \frac{2u}{(r_1+u_s)}\Big] \quad (2.13)$$

$$w'''' + \alpha_3 w_{tt} + c_3 w_t - \alpha_3[\alpha_1 \Gamma(u,u) + 2\alpha_1 \Gamma(u_s,u) + \alpha_2 \Gamma(w,w) + 2\alpha_2 \Gamma(w_s,w)] \\ w_s'' - \alpha_3[N + \alpha_1 \Gamma(u_s,u_s) + \alpha_1 \Gamma(u,u) + 2\alpha_1 \Gamma(u_s,u) + \alpha_2 \Gamma(w_s,w_s) + \alpha_2 \Gamma(w,w) \\ +2\alpha_2 \Gamma(w_s,w)]w'' = (2\beta_g + \beta_{3g}(1-w_s)^3)w\frac{(V_{1g})^2}{(1-w_s)^3} + \beta_g \frac{2V_{1g}v(t)+v(t)^2}{(1-w_s)^2} \\ \Big[1 + \frac{2w}{(1-w_s)}\Big] - \beta_{2g}(2V_{1g}v(t)+v(t)^2) - \beta_{3g}(2V_{1g}v(t)+v(t)^2)(1-w_s \\ -w) - \alpha_{2g}w[(V_{10})^2 + (V_{12})^2] - (2V_{10}v(t)+v(t)^2)(\alpha_g - \alpha_{2g}w_s - \alpha_{2g}w) \\ -(2V_{12}v(t)+v(t)^2)(\alpha_g - \alpha_{2g}w_s - \alpha_{2g}w) \quad (2.14)$$

2.2 Reduced order model

To obtain the static and dynamic equations to perform coupled analysis under 1:1 internal resonance condition near the coupling region which is much below the pull-in voltage [3], we use Galerkin method based on first mode approximation of the in-plane and out-of-plane displacements with negligible error [26]. Assuming the static and dynamic displacements subjected to the first transverse mode $\phi(x)$, the displacement along in-plane and out-of-plane directions can be written as [2, 3]:

$$u_s(x) = q_1(y,z)\phi(x), \quad w_s(x) = q_2(y,z)\phi(x), \quad (2.15)$$

$$u(x,t) = P_1(t)\phi(x), \quad w(x,t) = P_2(t)\phi(x), \quad (2.16)$$

where, q_1 and q_2 are static deflections, and $P_1(t)$ and $P_2(t)$ are non-dimensional modal variables. $\phi(x)$ is undamped exact mode shape [2] which is taken as $\phi(x) = \cosh(\zeta x) - \cos(\zeta x) - \nu\Big(\sinh(\zeta x) - \sin(\zeta x)\Big)$, where, for the first mode $\zeta = 4.73$, and $\nu = 0.9825$ such that $\int_0^1 (\phi_1(x))^2 dx = 1$. After premultiplying the denominator terms on either side of the Eqns (2.11), (2.12), substituting the assumed solution given by Eqn. (2.15) and applying Galerkin's method, we obtain the nonlinear static equations in both the directions which are given in Appendix A.

Similarly, by premultiplying the denominator terms on either side of the Eqns. (2.13) and (2.14), substituting the assumed static and dynamic solutions given by Eqns. (2.15) and (2.16), and applying Galerkin's method, we obtain the nonlinear modal dynamic equations in both the directions as:

$$
\begin{aligned}
P_{1tt}(t) + \lambda_1^2 P_1(t) + t_1 P_1(t)^3 + t_2 P_1(t)^2 + \big[t_3 P_2(t)^2 + t_4 P_2(t) + t_5(2V_{10}v(t) \\
+v(t)^2) + t_6(2V_{12}v(t) + v(t)^2)\big] P_1(t) + t_{11} P_{1t}(t) + t_7 P_2(t)^2 + t_8 P_2(t) \\
+t_9(2V_{10}v(t) + v(t)^2) + t_{10}(2V_{12}v(t) + v(t)^2) = 0 \quad (2.17) \\
P_{2tt}(t) + \lambda_2^2 P_2(t) + s_1 P_2(t)^3 + s_2 P_2(t)^2 + \big[s_3 P_2(t)^2 + s_4 P_2(t) + s_5(2V_{1g}v(t) \\
+v(t)^2) + s_6(2V_{10}v(t) + 2V_{12}v(t) + 2v(t)^2)\big] P_1(t) + s_{11} P_{2t}(t) + s_7 P_1(t)^2 \\
+s_8 P_1(t) + s_9(2V_{1g}v(t) + v(t)^2) + s_{10}(2V_{10}v(t) + 2V_{12}v(t) + 2v(t)^2) = 0 \quad (2.18)
\end{aligned}
$$

where, all coefficients of each term in above Eqns. (2.17) and (2.18) are given in Appendix A. Neglecting the damping term, nonlinear terms and the dynamic forcing terms from above Eqns. (2.17) and (2.18), we get linear modal dynamic equations as:

$$P_{1tt}(t) + \lambda_1^2 P_1(t) + t_8 P_2(t) = 0 \quad (2.19)$$

$$P_{2tt}(t) + \lambda_2^2 P_2(t) + s_8 P_1(t) = 0. \quad (2.20)$$

To obtain the linear frequency from the above equation, we assume the solution of Eqns.(2.19) and (2.20) as

$$P_1(t) = \beta e^{i\omega t}, \qquad P_2(t) = \gamma e^{i\omega t}.$$

Substituting the assumed solutions in the modal Eqns.(2.19) and (2.20), we get

$$
\begin{aligned}
(\lambda_1^2 - \omega^2)\beta + t_8\gamma = 0 \\
(\lambda_2^2 - \omega^2)\gamma + s_8\beta = 0
\end{aligned}
$$

For non-trivial solution, the determinant of these system of equations should be zero. After solving the resulting equation, we get two values of ω corresponding to two directions as

$$\omega_{1,2} = \sqrt{\frac{1}{2}\left[(\lambda_1^2 + \lambda_2^2) \pm \sqrt{(\lambda_1^2 - \lambda_2^2)^2 + 4t_8 s_8}\right]} \quad (2.21)$$

where, λ_1, λ_2, t_8 and s_8 are given in Appendix A.

3 Method of multiple scale

In this section, we apply the method of multiple scales (MMS) in solving Eqns. (2.17) and (2.18) assuming the modal displacements as functions multiple time scales, $T_0 = t$, $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$ as

$$
\begin{aligned}
P_1(t) = \epsilon x_{11}(T_0, T_1, T_2) + \epsilon^2 x_{12}(T_0, T_1, T_2) + \epsilon^3 x_{13}(T_0, T_1, T_2) + O(\epsilon^4) \\
P_2(t) = \epsilon x_{21}(T_0, T_1, T_2) + \epsilon^2 x_{22}(T_0, T_1, T_2) + \epsilon^3 x_{23}(T_0, T_1, T_2) + O(\epsilon^4), \quad (3.1)
\end{aligned}
$$

where, ϵ is a dimensionless small positive number. Subsequently, the derivative terms with respect to t can be defined in terms of new time scales as,

$$\frac{d}{dt} = \frac{\partial}{\partial T_0}\frac{dT_0}{dt} + \frac{\partial}{\partial T_1}\frac{dT_1}{dt} + \frac{\partial}{\partial T_2}\frac{dT_2}{dt} = (D_0 + \epsilon D_1 + \epsilon^2 D_2)$$

$$\frac{d^2}{dt^2} = \left(\frac{d}{dt}\right)^2 = (D_0 + \epsilon D_1 + \epsilon^2 D_2)^2 = (D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2 + 2\epsilon^2 D_0 D_2) + H.O.T \quad (3.2)$$

Rescaling the damping and forcing terms with different powers of ϵ as

$$t_{11} = \epsilon t_{11} \quad , \quad s_{11} = \epsilon s_{11} \quad , \quad v(t) = \epsilon^2 V_{ac}\cos(\Omega t), \quad (3.3)$$

substituting the assumed solution from Eqns. (3.1) to (3.3) in Eqns (2.17) and (2.18), and by comparing different powers of ϵ upto third order, we get the following three sets of equations as

$$O(\epsilon^1) \rightarrow \; D_0^2 x_{11} + \lambda_1^2 x_{11} + t_8 x_{21} = 0$$
$$D_0^2 x_{21} + \lambda_2^2 x_{21} + s_8 x_{11} = 0 \quad (3.4)$$

$$O(\epsilon^2) \rightarrow \; D_0^2 x_{12} + \lambda_1^2 x_{12} + t_8 x_{22} = -2D_0 D_1 x_{11} - t_{11} D_0 x_{11} - t_2 x_{11}^2 - t_4 x_{11} x_{21} - t_7 x_{21}^2 - t_9 \eta_{11}\cos(\omega_{ac} t) - t_{10}\eta_{12}\cos(\omega_{ac} t)$$
$$D_0^2 x_{22} + \lambda_2^2 x_{22} + s_8 x_{12} = -2D_0 D_1 x_{21} - s_{11} D_0 x_{21} - s_2 x_{21}^2 - s_4 x_{11} x_{21} - s_7 x_{11}^2 - s_9 \eta_{21}\cos(\omega_{ac} t) - s_{10}(\eta_{11} + \eta_{12})\cos(\omega_{ac} t) \quad (3.5)$$

$$O(\epsilon^3) \rightarrow \; D_0^2 x_{13} + \lambda_1^2 x_{13} + t_8 x_{23} = -t_{11}(D_0 x_{12} + D_1 x_{11}) - (2D_0 D_1 x_{12} + D_1^2 x_{11} + 2D_0 D_2 x_{11}) - t_1 x_{11}^3 - 2t_2 x_{11} x_{12} - t_3 x_{11} x_{21}^2 - t_4(x_{11} x_{22} + x_{12} x_{21}) - t_5 \eta_{11}\cos(\omega_{ac} t) x_{11} - t_6 \eta_{12}\cos(\omega_{ac} t) x_{11} - 2t_7 x_{21} x_{22}$$
$$D_0^2 x_{23} + \lambda_2^2 x_{23} + s_8 x_{13} = -s_{11}(D_0 x_{22} + D_1 x_{21}) - (2D_0 D_1 x_{22} + D_1^2 x_{21} + 2D_0 D_2 x_{21}) - s_1 x_{21}^3 - 2s_2 x_{21} x_{22} - s_3 x_{21} x_{11}^2 - s_4(x_{11} x_{22} + x_{12} x_{21}) - s_5 \eta_{21}\cos(\omega_{ac} t) x_{21} - s_6(\eta_{11} + \eta_{12})\cos(\omega_{ac} t) x_{21} - 2s_7 x_{11} x_{12} \quad (3.6)$$

where, $\eta_{11} = 2V_{10}V_{ac}$, $\eta_{12} = 2V_{12}V_{ac}$ and $\eta_{21} = 2V_{1g}V_{ac}$.

3.1 Solution of 1^{st} order equation

Solutions x_{11} and x_{21} for the two homogeneous second order coupled equations given by Eqn. (3.4) are can be written as

$$x_{11} = A_1(T_1,T_2)e^{i\omega_1 T_0} + A_2(T_1,T_2)e^{i\omega_2 T_0} + \bar{A}_1(T_1,T_2)e^{-i\omega_1 T_0} + \bar{A}_2(T_1,T_2)e^{-i\omega_2 T_0}$$

$$x_{21} = k_1 A_1(T_1,T_2)e^{i\omega_1 T_0} + k_2 A_2(T_1,T_2)e^{i\omega_2 T_0} + k_1 \bar{A}_1(T_1,T_2)e^{-i\omega_1 T_0} + k_2 \bar{A}_2(T_1,T_2)e^{-i\omega_2 T_0} \quad (3.7)$$

where, ω_1 and ω_2 are the coupled natural frequencies of the system in two orthogonal directions obtained from linear analysis. Substituting the assumed form of the solution from Eqn.(3.7) into Eqn.(3.4), we get

$$\begin{aligned}[(\lambda_1^2-\omega_1^2)+t_8k_1]A_1e^{i\omega_1T_0}+[(\lambda_1^2-\omega_2^2)+t_8k_2]A_2e^{i\omega_2T_0}=0\\ [(\lambda_2^2-\omega_1^2)k_1+s_8]A_1e^{i\omega_1T_0}+[(\lambda_2^2-\omega_2^2)k_2+s_8]A_2e^{i\omega_2T_0}=0.\end{aligned}\tag{3.8}$$

Equating the coefficients of different terms on both sides of Eqn.(3.8), we get

$$(\lambda_1^2-\omega_1^2)+t_8k_1=0,\quad (\lambda_1^2-\omega_2^2)+t_8k_2=0,\tag{3.9}$$

$$(\lambda_2^2-\omega_1^2)k_1+s_8=0,\quad (\lambda_2^2-\omega_2^2)k_2+s_8=0.\tag{3.10}$$

On solving Eqns.(3.9) and (3.10) for k_1 and k_2, we get the solvability condition as

$$k_n=\left(\frac{\omega_n^2-\lambda_1^2}{t_8}\right)=\left(\frac{s_8}{\omega_n^2-\lambda_2^2}\right),\tag{3.11}$$

where, $n=1$ and 2.

3.2 Solution of 2$^{\text{nd}}$ order equation

To obtain the solution of Eqn. (3.5), we substitute Eqn. (3.7) into it and eliminate the secular terms. Taking the detuning parameters σ_1 and σ_2 as

$$\Omega=\omega_1+\epsilon\sigma_1,\quad \omega_2=\omega_1+\epsilon\sigma_2.\tag{3.12}$$

and substituting x_{11} and x_{21} from Eqn. (3.7) into Eqn. (3.5), we get

$$\begin{aligned}D_0^2x_{12}+\lambda_1^2x_{12}+t_8x_{22}=&-2i(\omega_1D_1A_1e^{i\omega_1T_0}+\omega_2D_1A_2e^{i\omega_2T_0})-it_{11}(A_1\omega_1e^{i\omega_1T_0}\\ &+A_2\omega_2e^{i\omega_2T_0})-t_2(A_1^2e^{2i\omega_1T_0}+A_1\bar{A}_1+A_2^2e^{2i\omega_2T_0}+A_2\bar{A}_2+2A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0}\\ &+2A_1A_2e^{i(\omega_1+\omega_2)T_0})-t_4(A_1^2k_1e^{2i\omega_1T_0}+A_1\bar{A}_1k_1+A_2^2k_2e^{2i\omega_2T_0}+A_2\bar{A}_2k_2+A_1\bar{A}_2\\ &(k_1+k_2)e^{i(\omega_1-\omega_2)T_0}+A_1A_2(k_1+k_2)e^{i(\omega_1+\omega_2)T_0})-t_7(k_1^2A_1^2e^{2i\omega_1T_0}+k_1^2A_1\bar{A}_1\\ &+k_2^2A_2^2e^{2i\omega_2T_0}+k_2^2A_2\bar{A}_2+2k_1k_2A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0}+2k_1k_2A_1A_2e^{i(\omega_1+\omega_2)T_0})\\ &+\frac{t_9}{2}\eta_{11}e^{i\omega_{ac}T_0}-\frac{t_{10}}{2}\eta_{12}e^{i\omega_{ac}T_0}+cc\end{aligned}\tag{3.13}$$

$$\begin{aligned}D_0^2x_{22}+\lambda_2^2x_{22}+s_8x_{12}=&-2i(\omega_1k_1D_1A_1e^{i\omega_1T_0}+k_2\omega_2D_1A_2e^{i\omega_2T_0})-is_{11}(k_1A_1\\ &\omega_1e^{i\omega_1T_0}+k_2A_2\omega_2e^{i\omega_2T_0})-s_2(k_1^2A_1^2e^{2i\omega_1T_0}+k_1^2A_1\bar{A}_1+k_2^2A_2^2e^{2i\omega_2T_0}+k_2^2A_2\bar{A}_2\\ &+2k_1k_2A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0}+2k_1k_2A_1A_2e^{i(\omega_1+\omega_2)T_0})-s_4(A_1^2k_1e^{2i\omega_1T_0}+A_1\bar{A}_1k_1\\ &+A_2^2k_2e^{2i\omega_2T_0}+A_2\bar{A}_2k_2+A_1\bar{A}_2(k_1+k_2)e^{i(\omega_1-\omega_2)T_0}+A_1A_2(k_1+k_2)e^{i(\omega_1+\omega_2)T_0})\\ &-s_7(A_1^2e^{2i\omega_1T_0}+A_1\bar{A}_1+A_2^2e^{2i\omega_2T_0}+A_2\bar{A}_2+2A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0}+2A_1A_2\\ &e^{i(\omega_1+\omega_2)T_0})+\frac{s_9}{2}\eta_{21}e^{i\omega_{ac}T_0}-\frac{s_{10}}{2}(\eta_{11}+\eta_{12})e^{i\omega_{ac}T_0}+cc.\end{aligned}\tag{3.14}$$

Assuming the solution of homogeneous form of Eqn.(3.5) as

$$\begin{aligned}x_{12}&=P_{11}e^{i\omega_1T_0}+P_{12}e^{i\omega_2T_0}+cc,\\ x_{22}&=P_{21}e^{i\omega_1T_0}+P_{22}e^{i\omega_2T_0}+cc\end{aligned}\tag{3.15}$$

and substituting it in Eqns. (3.13) and (3.14), and eliminating the secular terms, we get

$$(\lambda_1^2-\omega_1^2)P_{11}e^{i\omega_1T_0}+(\lambda_1^2-\omega_2^2)P_{12}e^{i\omega_2T_0}+t_8(P_{21}e^{i\omega_1T_0}+P_{22}e^{i\omega_2T_0})$$
$$=R_{11}e^{i\omega_1T_0}+R_{12}e^{i\omega_2T_0} \quad (3.16)$$
$$(\lambda_2^2-\omega_1^2)P_{21}e^{i\omega_1T_0}+(\lambda_2^2-\omega_2^2)P_{22}e^{i\omega_2T_0}+s_8(P_{11}e^{i\omega_1T_0}+P_{12}e^{i\omega_2T_0})$$
$$=R_{21}e^{i\omega_1T_0}+R_{22}e^{i\omega_2T_0}. \quad (3.17)$$

The above equations can also be written in the matrix form as

$$\begin{bmatrix}(\lambda_1^2-\omega_n^2) & t_8\\ s_8 & (\lambda_2^2-\omega_n^2)\end{bmatrix}\begin{bmatrix}P_{1n}\\P_{2n}\end{bmatrix}=\begin{bmatrix}R_{1n}\\R_{2n}\end{bmatrix}$$

where, $n=1$ and 2, R_{1n} and R_{2n} are the coefficients of $e^{i\omega_1T_0}$ and $e^{i\omega_2T_0}$ appearing in Eqns. (3.13) and (3.14) as

$$R_{11}=-2i\omega_1D_1A_1-2i\omega_2D_1A_2e^{i\sigma_2T_1}-it_{11}A_1\omega_1-it_{11}A_2\omega_2e^{i\sigma_2T_1}$$
$$-\left(\frac{t_9\eta_{11}+t_{10}\eta_{12}}{2}\right)e^{i\sigma_1T_1}$$

$$R_{12}=-2i\omega_1D_1A_1e^{-i\sigma_2T_1}-2i\omega_2D_1A_2-it_{11}A_1\omega_1e^{-i\sigma_2T_1}-it_{11}A_2\omega_2$$
$$-\left(\frac{t_9\eta_{11}+t_{10}\eta_{12}}{2}\right)e^{i(\sigma_1-\sigma_2)T_1}$$

$$R_{21}=-2i\omega_1k_1D_1A_1-2ik_2\omega_2D_1A_2e^{i\sigma_2T_1}-is_{11}k_1A_1\omega_1-is_{11}k_2A_2\omega_2e^{i\sigma_2T_1}$$
$$-\left(\frac{s_9\eta_{21}+s_{10}(\eta_{11}+\eta_{12})}{2}\right)e^{i\sigma_1T_1}$$

$$R_{22}=-2i\omega_1k_1D_1A_1e^{-i\sigma_2T_1}-2i\omega_2k_2D_1A_2-is_{11}k_1A_1\omega_1e^{-i\sigma_2T_1}-is_{11}k_2A_2\omega_2$$
$$-\left(\frac{s_9\eta_{21}+s_{10}(\eta_{11}+\eta_{12})}{2}\right)e^{i(\sigma_1-\sigma_2)T_1}. \quad (3.18)$$

Solving Eqns. (3.16) and (3.17), and using Eqn. (3.11), we get the solvability conditions in terms of R_{1n} and R_{2n} as

$$R_{1n}=\frac{t_8}{(\lambda_2^2-\omega_n^2)}R_{2n}=-\left(\frac{t_8}{s_8}\right)k_nR_{2n}=-\bar{k_n}R_{2n}, \quad (3.19)$$

where, $\bar{k_n}=(\frac{t_8}{s_8})k_n$. For $n=1$ and $\bar{k_1}=(\frac{t_8}{s_8})k_1$, the solvability condition $R_{11}+\bar{k_1}R_{21}=0$ reduces to

$$-2i\omega_1(1+k_1\bar{k_1})D_1A_1-2i\omega_2(1+k_2\bar{k_1})D_1A_2e^{i\sigma_2T_1}$$
$$=i\omega_1A_1(t_{11}+s_{11}k_1\bar{k_1})+i\omega_2A_2(t_{11}+s_{11}k_2\bar{k_1})e^{i\sigma_2T_1}$$
$$+\left(\frac{t_9\eta_{11}+t_{10}\eta_{12}+\bar{k_1}s_9\eta_{21}+\bar{k_1}s_{10}(\eta_{11}+\eta_{12})}{2}\right)e^{i\sigma_1T_1}. \quad (3.20)$$

Similarly, for n =2 and $\bar{k_2} = (\frac{t_8}{s_8})k_2$, the solvability condition $R_{12} + \bar{k_2}R_{22} = 0$ can be written as

$$\begin{aligned}
-2i\omega_1(1 + k_1\bar{k_2})D_1A_1e^{-i\sigma_2 T_1} - 2i\omega_2(1 + k_2\bar{k_2})D_1A_2 \\
= i\omega_2A_2(t_{11} + s_{11}k_2\bar{k_2}) + i\omega_1A_1(t_{11} + s_{11}k_1\bar{k_2})e^{-i\sigma_2 T_1} \\
+\Big(\frac{t_9\eta_{11} + t_{10}\eta_{12} + \bar{k_2}s_9\eta_{21} + \bar{k_2}s_{10}(\eta_{11} + \eta_{12})}{2}\Big)e^{i(\sigma_1-\sigma_2)T_1}.
\end{aligned} \tag{3.21}$$

Solving Eqns. (3.20) and (3.21), simultaneously, we find the expressions for D_1A_1 and D_1A_2 as,

$$D_1A_1 = \frac{B_1A_1 + B_2A_2e^{i\sigma_2 T_1} + iB_3e^{i\sigma_1 T_1}}{B_4}$$

$$D_1A_2 = \frac{B_5A_2 + B_6A_1e^{-i\sigma_2 T_1} + iB_7e^{i(\sigma_1-\sigma_2)T_1}}{B_8} \tag{3.22}$$

where, $B_1, B_2, ..., B_8$ are defined in Appendix B. Now, substituting D_1A_1 and D_1A_2 from Eqn.(3.22) into the second order Eqns. (3.13) and (3.14), we obtain the complete solution for second order equations as

$$\begin{aligned}
x_{12} = A_3(T_1, T_2)e^{i\omega_1 T_0} + A_4(T_1, T_2)e^{i\omega_2 T_0} + c_{11}A_1^2e^{2i\omega_1 T_0} + c_{12}A_1\bar{A_1} + c_{13}A_2^2 \\
e^{2i\omega_2 T_0} + c_{14}A_2\bar{A_2} + c_{15}A_1\bar{A_2}e^{i(\omega_1-\omega_2)T_0} + c_{16}A_1A_2e^{i(\omega_1+\omega_2)T_0} + cc
\end{aligned}$$

$$\begin{aligned}
x_{22} = k_1A_3(T_1, T_2)e^{i\omega_1 T_0} + k_2A_4(T_1, T_2)e^{i\omega_2 T_0} + c_{21}A_1^2e^{2i\omega_1 T_0} + c_{22}A_1\bar{A_1} + c_{23}A_2^2 \\
e^{2i\omega_2 T_0} + c_{24}A_2\bar{A_2} + c_{25}A_1\bar{A_2}e^{i(\omega_1-\omega_2)T_0} + c_{26}A_1A_2e^{i(\omega_1+\omega_2)T_0} + cc
\end{aligned} \tag{3.23}$$

On substituting Eqn.(3.23) in Eqns. (3.13) and (3.14), and comparing the coefficients of same terms on both side of the resulting equations, we obtain the following equations in terms of coefficients c_{ij}.

Coefficients of $A_1^2e^{2i\omega_1 T_0}$:

$$\begin{aligned}
-4c_{11}\omega_1^2 + c_{11}\lambda_1^2 + t_8c_{21} = -t_2 - t_4k_1 - t_7k_1^2 \\
-4c_{21}\omega_1^2 + c_{21}\lambda_2^2 + s_8c_{11} = -s_7 - s_4k_1 - s_2k_1^2
\end{aligned} \tag{3.24}$$

Coefficients of $A_1\bar{A_1}$:

$$\begin{aligned}
\lambda_1^2c_{12} + t_8c_{22} = -2t_2 - 2t_4k_1 - 2t_7k_1^2 \\
\lambda_2^2c_{22} + s_8c_{12} = -2s_7 - 2s_4k_1 - 2s_2k_1^2
\end{aligned} \tag{3.25}$$

Coefficients of $A_2^2e^{2i\omega_2 T_0}$:

$$\begin{aligned}
-4c_{13}\omega_2^2 + c_{13}\lambda_1^2 + t_8c_{23} = -t_2 - t_4k_2 - t_7k_2^2 \\
-4c_{23}\omega_2^2 + c_{23}\lambda_2^2 + s_8c_{13} = -s_7 - s_4k_2 - s_2k_2^2
\end{aligned} \tag{3.26}$$

Coefficients of $A_2\bar{A_2}$:

$$\begin{aligned}
\lambda_1^2c_{14} + t_8c_{24} = -2t_2 - 2t_4k_2 - 2t_7k_2^2 \\
\lambda_2^2c_{24} + s_8c_{14} = -2s_7 - 2s_4k_2 - 2s_2k_2^2
\end{aligned} \tag{3.27}$$

Coefficients of $A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0}$:

$$-c_{15}(\omega_1-\omega_2)^2+\lambda_1^2c_{15}+t_8c_{25}=-2t_2-t_4(k_1+k_2)-2t_7k_1k_2$$
$$-c_{25}(\omega_1-\omega_2)^2+\lambda_2^2c_{25}+s_8c_{15}=-2s_7-s_4(k_1+k_2)-2s_2k_1k_2 \quad (3.28)$$

Coefficients of $A_1A_2e^{i(\omega_1+\omega_2)T_0}$:

$$-c_{16}(\omega_1+\omega_2)^2+\lambda_1^2c_{16}+t_8c_{26}=-2t_2-t_4(k_1+k_2)-2t_7k_1k_2$$
$$-c_{26}(\omega_1+\omega_2)^2+\lambda_2^2c_{26}+s_8c_{16}=-2s_7-s_4(k_1+k_2)-2s_2k_1k_2 \quad (3.29)$$

Solving above equations simultaneously, the coefficients c_{ij} are obtained which are given in Appendix B.

3.3 Solution of 3$^{\text{rd}}$ order equation

Substituting Eqns. (3.7) and (3.23) into Eqn.(3.6), using $\Omega=\omega_1+\epsilon\sigma_1$, $\omega_2=\omega_1+\epsilon\sigma_2$ and separating the secular terms similar to previous section, we obtain the following equations:

$$R_{11}=-2i\omega_1(D_1A_3+D_2A_1)-D_1^2A_1-t_{11}D_1A_1-[2i\omega_2(D_1A_4+D_2A_2)+D_1^2A_2$$
$$+t_{11}D_1A_2]e^{i\sigma_2T_1}-it_{11}\omega_1A_3-it_{11}\omega_2A_4e^{i\sigma_2T_1}-g_{11}A_1^2\bar{A}_2e^{-i\sigma_2T_1}-g_{12}A_2^2\bar{A}_2e^{i\sigma_2T_1}$$
$$-g_{13}A_2A_1\bar{A}_1e^{i\sigma_2T_1}-g_{14}A_2^2\bar{A}_1e^{2i\sigma_2T_1}-g_{15}A_1^2\bar{A}_1-g_{16}A_1A_2\bar{A}_2 \quad (3.30)$$

$$R_{12}=-2i\omega_2(D_1A_4+D_2A_2)-D_2^2A_2-t_{11}D_1A_2-[2i\omega_1(D_1A_3+D_2A_1)+D_1^2A_1$$
$$-t_{11}D_1A_1]e^{-i\sigma_2T_1}-it_{11}\omega_2A_4-it_{11}\omega_1A_3e^{-i\sigma_2T_1}-f_{11}A_2^2\bar{A}_1e^{i\sigma_2T_1}-f_{12}A_1^2\bar{A}_1$$
$$e^{-i\sigma_2T_1}-f_{13}A_1A_2\bar{A}_2e^{-i\sigma_2T_1}-f_{14}A_1^2\bar{A}_2e^{-2i\sigma_2T_1}-f_{15}A_2^2\bar{A}_2-f_{16}A_2A_1\bar{A}_1 \quad (3.31)$$

$$R_{21}=-2i\omega_1k_1(D_1A_3+D_2A_1)-k_1D_1^2A_1-s_{11}k_1D_1A_1-[2ik_2\omega_2(D_1A_4+D_2A_2)$$
$$+k_2D_1^2A_2+k_2s_{11}D_1A_2]e^{i\sigma_2T_1}-is_{11}k_1\omega_1A_3-is_{11}k_2\omega_2A_4e^{i\sigma_2T_1}-g_{21}A_1^2\bar{A}_1e^{-i\sigma_2T_1}$$
$$-(g_{22}A_2^2\bar{A}_2+g_{23}A_2A_1\bar{A}_1)e^{i\sigma_2T_1}-g_{24}A_2^2\bar{A}_1e^{2i\sigma_2T_1}-g_{25}A_1^2\bar{A}_1-g_{26}A_1A_2\bar{A}_2 \quad (3.32)$$

$$R_{22}=-2ik_2\omega_2(D_1A_4+D_2A_2)-k_2D_1^2A_2-s_{11}k_2D_1A_2-[2ik_1\omega_1(D_1A_3+D_2A_1)$$
$$+k_1D_1^2A_1+s_{11}k_1D_1A_1]e^{-i\sigma_2T_1}-ik_2s_{11}\omega_2A_4-ik_1s_{11}\omega_1A_3e^{-i\sigma_2T_1}-f_{21}A_2^2\bar{A}_1$$
$$e^{i\sigma_2T_1}-f_{22}A_1^2\bar{A}_1e^{-i\sigma_2T_1}-f_{23}A_1A_2\bar{A}_2e^{-i\sigma_2T_1}-f_{24}A_1^2\bar{A}_2e^{-2i\sigma_2T_1}$$
$$-f_{25}A_2^2\bar{A}_2-f_{26}A_2A_1\bar{A}_1 \quad (3.33)$$

where, g_{n1},...,g_{n6} and f_{n1},...,f_{n6} for $n=1$ and 2 are given in Appendix B. Substituting the above equations in solvability condition given by Eqn.(3.19), we get the following two conditions

$$-2i\omega_1B_{11}(D_1A_3+D_2A_1)-B_{11}D_1^2A_1-B_{13}D_1A_1-[2i\omega_2B_{12}(D_1A_4+D_2A_2)$$
$$+B_{12}D_1^2A_2+B_{14}D_1A_2]e^{i\sigma_2T_1}=i\omega_1B_{13}A_3+i\omega_2B_{14}A_4e^{i\sigma_2T_1}+\bar{g_1}\bar{A}_2A_1^2e^{-i\sigma_2T_1}$$
$$+\bar{g_2}\bar{A}_2A_2^2e^{i\sigma_2T_1}+\bar{g_3}A_1A_2\bar{A}_1e^{i\sigma_2T_1}+\bar{g_4}A_2^2\bar{A}_1e^{2i\sigma_2T_1}+\bar{g_5}A_1^2\bar{A}_1+\bar{g_6}A_1\bar{A}_2A_2 \quad (3.34)$$

$$-2i\omega_2 G_{12}(D_1A_4 + D_2A_2) - G_{12}D_1^2A_2 - G_{13}D_1A_2 - [2i\omega_1 G_{11}(D_1A_3 + D_2A_1) + G_{11}D_1^2A_1 + G_{14}D_1A_1]e^{-i\sigma_2T_1} = i\omega_2 G_{13}A_4 + i\omega_1 G_{14}A_3e^{-i\sigma_2T_1} + \bar{f}_1\bar{A}_1A_2^2e^{i\sigma_2T_1} + \bar{f}_2\bar{A}_1A_1^2e^{-i\sigma_2T_1} + \bar{f}_3A_1A_2\bar{A}_2e^{-i\sigma_2T_1} + \bar{f}_4A_1^2\bar{A}_2e^{-2i\sigma_2T_1} + \bar{f}_5A_2^2\bar{A}_2 + \bar{f}_6A_1\bar{A}_1A_2, \tag{3.35}$$

where, $\bar{g_n} = g_{1n} + \bar{k_1}g_{2n}$, $\bar{f_n} = f_{1n} + \bar{k_2}f_{2n}$, and the coefficients B_{11}, B_{12}, B_{13}, B_{14}, G_{11}, G_{12}, G_{13}, G_{14}, f_{1n} and g_{1n} for $n = 1, ..., 6$ are mentioned in Appendix B.

Now, by following the procedure as mentioned in [27,17], we find A_3 and A_4 so as to eliminate $D_1^2A_1$ and $D_1^2A_2$ from Eqns. (3.34) and (3.35). Such conditions lead to the following equations,

$$D_1[2i\omega_1A_3 + D_1A_1] = 0 \quad \Rightarrow [2i\omega_1A_3 + D_1A_1] = h_{11}(T_2),$$

$$D_1[2i\omega_2A_4 + D_1A_2] = 0 \quad \Rightarrow [2i\omega_2A_4 + D_1A_2] = h_{12}(T_2); \tag{3.36}$$

However, it is evident from Eqn.(3.22) that D_1A_1 and D_1A_2 are implicit functions of slow time scale T_2. Thus, we take $h_{11}(T_2) = h_{12}(T_2)$ =0. Now, using Eqns.(3.22) and (3.36), the expressions for A_3 and A_4 can be written as,

$$A_3 = \frac{iB_1}{2\omega_1B_4}A_1 + \frac{iB_2}{2\omega_1B_4}A_2e^{i\sigma_2T_1} - \frac{B_3}{2\omega_1B_4}e^{i\sigma_1T_1}$$
$$A_4 = \frac{iB_5}{2\omega_2B_8}A_2 + \frac{iB_6}{2\omega_2B_8}A_1e^{-i\sigma_2T_1} - \frac{B_7}{2\omega_2B_8}e^{i(\sigma_1-\sigma_2)T_1}. \tag{3.37}$$

Using Eqn. (3.37) in Eqns. (3.34) and (3.35), we can find the values for D_2A_1 and D_2A_2 by solving the following equations,

$$-2i\omega_1B_{11}D_2A_1 - 2i\omega_2B_{12}D_2A_2e^{i\sigma_2T_1} = B_{13}D_1A_1 + B_{14}D_1A_2e^{i\sigma_2T_1} + i\omega_1B_{13}A_3 + i\omega_2B_{14}A_4e^{i\sigma_2T_1} + \bar{g_1}\bar{A}_2A_1^2e^{-i\sigma_2T_1} + \bar{g_2}\bar{A}_2A_2^2e^{i\sigma_2T_1} + \bar{g_3}A_1A_2\bar{A}_1e^{i\sigma_2T_1} + \bar{g_4}A_2^2\bar{A}_1e^{2i\sigma_2T_1} + \bar{g_5}A_1^2\bar{A}_1 + \bar{g_6}A_1\bar{A}_2A_2 \tag{3.38}$$

$$-2i\omega_1G_{11}D_2A_1e^{-i\sigma_2T_1} - 2i\omega_2G_{12}D_2A_2 = G_{13}D_1A_2 + G_{14}D_1A_1e^{-i\sigma_2T_1} + i\omega_2G_{13}A_4 + i\omega_1G_{14}A_3e^{-i\sigma_2T_1} + \bar{f}_1\bar{A}_1A_2^2e^{i\sigma_2T_1} + \bar{f}_2\bar{A}_1A_1^2e^{-i\sigma_2T_1} + \bar{f}_3A_1A_2\bar{A}_2e^{-i\sigma_2T_1} + \bar{f}_4A_1^2\bar{A}_2e^{-2i\sigma_2T_1} + \bar{f}_5A_2^2\bar{A}_2 + \bar{f}_6A_1\bar{A}_1A_2. \tag{3.39}$$

The expressions for D_2A_1 and D_2A_2 can be obtained as

$$D_2A_1 = \frac{i}{2\omega_1(B_{11}G_{12} - B_{12}G_{11})}\Big[(B_{14}G_{12} - B_{12}G_{13})D_1A_2e^{i\sigma_2T_1} + (B_{13}G_{12} - B_{12}G_{14})D_1A_1 + i\omega_1(B_{13}G_{12} - B_{12}G_{14})A_3 + i\omega_2(B_{14}G_{12} - B_{12}G_{13})A_4e^{i\sigma_2T_1} + G_{12}\bar{G} - B_{12}\bar{F}e^{i\sigma_2T_1}\Big],$$

$$D_2A_2 = \frac{i}{2\omega_2(B_{12}G_{11} - B_{11}G_{12})}\Big[(B_{14}G_{11} - B_{11}G_{13})D_1A_2 + (B_{13}G_{11} - B_{11}G_{14})D_1A_1e^{-i\sigma_2T_1} + i\omega_2(B_{14}G_{11} - B_{11}G_{13})A_4 + i\omega_1(B_{13}G_{11} - B_{11}G_{14})A_3e^{-i\sigma_2T_1} + G_{11}\bar{G}e^{-i\sigma_2T_1} - B_{11}\bar{F}\Big] \tag{3.40}$$

where, $\bar{F}$ and $\bar{G}$ are given in the Appendix B. To get the final solution, we apply the method of reconstitution [27,17] and write modulation in the following form

$$\begin{aligned}\frac{dA_1}{dt} &= \epsilon D_1 A_1 + \epsilon^2 D_2 A_1 + ...\\ \frac{dA_2}{dt} &= \epsilon D_1 A_2 + \epsilon^2 D_2 A_2 + ... \end{aligned} \tag{3.41}$$

Substituting the values of D_1A_1 and D_1A_2 from Eqn. (3.22), D_2A_1 and D_2A_2 from Eqn. (3.40), and A_3 and A_4 from Eqn.(3.37) into Eqn.(3.41), and setting $\epsilon = 1$ such that $T_0 = T_1 = T_2 = t$, we get the reconstituted modulation equations as

$$\begin{aligned}\dot{A_1} = &\left[1 + i\frac{(B_{13}G_{12} - G_{14}B_{12})}{4\omega_1(B_{11}G_{12} - G_{11}B_{12})}\right]\left[\frac{B_1}{B_4}A_1 + \frac{B_2}{B_4}A_2e^{i\sigma_2 t} + i\frac{B_3}{B_4}e^{i\sigma_1 t}\right]\\ &+ i\left[\frac{(B_{14}G_{12} - G_{13}B_{12})}{4\omega_1(B_{11}G_{12} - G_{11}B_{12})}\right]e^{i\sigma_2 t}\left[\frac{B_5}{B_8}A_2 + \frac{B_6}{B_8}A_1e^{-i\sigma_2 t} + i\frac{B_7}{B_8}e^{i(\sigma_1-\sigma_2)t}\right]\\ &+ i\left[\frac{(G_{12}\bar{G} - B_{12}\bar{F}e^{i\sigma_2 t})}{2\omega_1(B_{11}G_{12} - G_{11}B_{12})}\right],\end{aligned} \tag{3.42}$$

$$\begin{aligned}\dot{A_2} = &\, i\left[\frac{(B_{13}G_{11} - G_{14}B_{11})}{4\omega_2(B_{12}G_{11} - G_{12}B_{11})}\right]e^{-i\sigma_2 t}\left[\frac{B_1}{B_4}A_1 + \frac{B_2}{B_4}A_2e^{i\sigma_2 t} + i\frac{B_3}{B_4}e^{i\sigma_1 t}\right]\\ &+ \left[1 + i\frac{(B_{14}G_{11} - G_{13}B_{11})}{4\omega_2(B_{12}G_{11} - G_{12}B_{11})}\right]\left[\frac{B_5}{B_8}A_2 + \frac{B_6}{B_8}A_1e^{-i\sigma_2 t} + i\frac{B_7}{B_8}e^{i(\sigma_1-\sigma_2)t}\right]\\ &+ i\left[\frac{(G_{11}\bar{G}e^{-i\sigma_2 t} - B_{11}\bar{F})}{2\omega_2(B_{12}G_{11} - G_{12}B_{11})}\right].\end{aligned} \tag{3.43}$$

To express the modulation equations in polar form, we rewrite A_1 and A_2 as,

$$A_n = \frac{1}{2}a_n e^{i\beta_n}, \quad n = 1, 2. \tag{3.44}$$

Here, a_n and β_n are real functions of time t, hence, $\dot{A_1}$ and $\dot{A_2}$ can be written as

$$\begin{aligned}A_1 = \frac{1}{2}a_1e^{i\beta_1} \qquad &\Rightarrow \dot{A_1} = \frac{1}{2}(\dot{a_1}e^{i\beta_1} + ia_1\dot{\beta_1}e^{i\beta_1})\\ A_2 = \frac{1}{2}a_2e^{i\beta_2} \qquad &\Rightarrow \dot{A_2} = \frac{1}{2}(\dot{a_2}e^{i\beta_2} + ia_2\dot{\beta_2}e^{i\beta_2}).\end{aligned} \tag{3.45}$$

Substituting Eqn.(3.45) into Eqns.(3.42) and (3.43), we get

$$\begin{aligned}&\frac{1}{2}(\dot{a_1}e^{i\beta_1} + ia_1\dot{\beta_1}e^{i\beta_1}) = (1 + ih_{11})\left[h_{22}a_1e^{i\beta_1} + h_{33}a_2e^{i(\beta_2+\sigma_2 t)} + ih_{44}e^{i\sigma_1 t}\right]\\ &\quad +ih_{55}e^{i\sigma_2 t}\left[h_{66}a_2e^{i\beta_2} + h_{77}a_1e^{i(\beta_1-\sigma_2 t)} + ih_{88}e^{i(\sigma_1-\sigma_2)t}\right] + ih_{99}\Big[\bar{g_1}a_1^2a_2\\ &\quad e^{i(2\beta_1-\beta_2-\sigma_2 t)} + \bar{g_2}a_2^3e^{i(\beta_2+\sigma_2 t)} + \bar{g_3}a_1^2a_2e^{i(\beta_2+\sigma_2 t)} + \bar{g_4}a_1a_2^2e^{i(2\beta_2-\beta_1+2\sigma_2 t)}\\ &\quad +\bar{g_5}a_1^3e^{i\beta_1} + \bar{g_6}a_1a_2^2e^{i\beta_1}\Big] - ih_{1010}\Big[\bar{f_1}a_1a_2^2e^{i(2\sigma_2 t-\beta_1+2\beta_2)} + \bar{f_2}a_1^3e^{i\beta_1} + \bar{f_3}a_1\\ &\quad a_2^2e^{i\beta_1} + \bar{f_4}a_1^2a_2e^{i(2\beta_1-\beta_2-\sigma_2 t)} + \bar{f_5}a_2^3e^{i(\beta_2+\sigma_2 t)} + \bar{f_6}a_1^2a_2e^{i(\beta_2+\sigma_2 t)}\Big]\end{aligned} \tag{3.46}$$

$$\frac{1}{2}(\dot{a_2}e^{i\beta_2} + ia_2\dot{\beta_2}e^{i\beta_2}) = \big(1 + il_{11}\big)\big[l_{22}a_2e^{i\beta_2} + l_{33}a_1e^{i(\beta_1-\sigma_2 t)} + il_{44}e^{i(\sigma_1-\sigma_2)t}\big]$$
$$+il_{55}e^{-i\sigma_2 t}\big[l_{66}a_1e^{i\beta_1} + l_{77}a_2e^{i(\beta_2+\sigma_2 t)} + il_{88}e^{i\sigma_1 t}\big] + il_{99}\Big[\bar{g_1}a_1^2a_2e^{i(2\beta_1-\beta_2-2\sigma_2 t)}$$
$$+\bar{g_2}a_2^3e^{i\beta_2} + \bar{g_3}a_1^2a_2e^{i\beta_2} + \bar{g_4}a_1a_2^2e^{i(2\beta_2-\beta_1+\sigma_2 t)} + \bar{g_5}a_1^3e^{i(\beta_1-\sigma_2 t)} + \bar{g_6}a_1a_2^2$$
$$e^{i(\beta_1-\sigma_2 t)}\Big] - il_{1010}\Big[\bar{f_1}a_1a_2^2e^{i(\sigma_2 t-\beta_1+2\beta_2)} + \bar{f_2}a_1^3e^{i(\beta_1-\sigma_2 t)} + \bar{f_3}a_1a_2^2e^{i(\beta_1-\sigma_2 t)}$$
$$+\bar{f_4}a_1^2a_2e^{i(2\beta_1-\beta_2-2\sigma_2 t)} + \bar{f_5}a_2^3e^{i\beta_2} + \bar{f_6}a_1^2a_2e^{i\beta_2}\Big] \quad (3.47)$$

where, $h_{11},\ h_{22}, ..., h_{1010}$ and $l_{11},\ l_{22}, ..., l_{1010}$ are given in Appendix B. Finally, we convert the above non-autonomous equations into autonomous forms by defining two new variables and their corresponding time derivative terms as

$$\theta_1 = (\sigma_1 t - \beta_1), \quad \theta_2 = (\sigma_1 - \sigma_2)t - \beta_2,$$
$$\dot{\theta_1} = (\sigma_1 - \dot{\beta_1}), \quad \dot{\theta_2} = (\sigma_1 - \sigma_2) - \dot{\beta_2} \quad (3.48)$$

Substiting Eqn.(3.48) into Eqns.(3.46) and (3.47), and separating the real and imaginary parts, we get the following form of modulation equations

$$\dot{a_1} = h_2a_1 + h_3a_2\cos(\theta_1-\theta_2) - h_4\sin(\theta_1) + h_1[-h_3a_2\sin(\theta_1-\theta_2) - h_4\cos(\theta_1)]$$
$$+ h_5[-h_6a_2\sin(\theta_1-\theta_2) - h_8\cos(\theta_1)] + h_9[\bar{g_1}a_1^2a_2\sin(\theta_1-\theta_2) - \bar{g_2}a_2^3$$
$$\sin(\theta_1-\theta_2) - \bar{g_3}a_1^2a_2\sin(\theta_1-\theta_2) - \bar{g_4}a_2^2a_1\sin 2(\theta_1-\theta_2)] - h_{10}[-\bar{f_1}a_2^2a_1$$
$$\sin 2(\theta_1-\theta_2) + \bar{f_4}a_1^2a_2\sin(\theta_1-\theta_2) - \bar{f_5}a_2^3\sin(\theta_1-\theta_2)$$
$$- \bar{f_6}a_1^2a_2\sin(\theta_1-\theta_2)] \quad (3.49)$$

$$\dot{a_1}\dot{\theta_1} = a_1\sigma_1 - h_3a_2\sin(\theta_1-\theta_2) - h_4\cos(\theta_1) - h_1[h_2a_1 + h_3a_2\cos(\theta_1-\theta_2)$$
$$- h_4\sin(\theta_1)] - h_5[h_6a_2\cos(\theta_1-\theta_2) + h_7a_1 - h_8\sin(\theta_1)] - h_9[\bar{g_1}a_1^2a_2$$
$$\cos(\theta_1-\theta_2) + \bar{g_2}a_2^3\cos(\theta_1-\theta_2) + \bar{g_3}a_1^2a_2\cos(\theta_1-\theta_2) + \bar{g_4}a_2^2a_1$$
$$\cos 2(\theta_1-\theta_2) + \bar{g_5}a_1^3 + \bar{g_6}a_1a_2^2] + h_{10}[\bar{f_1}a_2^2a_1\cos 2(\theta_1-\theta_2) + \bar{f_2}a_1^3$$
$$+ \bar{f_3}a_1a_2^2 + \bar{f_4}a_1^2a_2\cos(\theta_1-\theta_2) + \bar{f_5}a_2^3\cos(\theta_1-\theta_2)$$
$$+ \bar{f_6}a_1^2a_2\cos(\theta_1-\theta_2)] \quad (3.50)$$

$$\dot{a_2} = l_2a_2 + l_3a_1\cos(\theta_1-\theta_2) - l_4\sin(\theta_2) + l_1[l_3a_1\sin(\theta_1-\theta_2) - l_4\cos(\theta_2)]$$
$$+ l_5[l_6a_1\sin(\theta_1-\theta_2) - l_8\cos(\theta_2)] + l_9[\bar{g_1}a_1^2a_2\sin 2(\theta_1-\theta_2) - \bar{g_4}a_2^2a_1$$
$$\sin(\theta_1-\theta_2) + \bar{g_5}a_1^3a_2\sin(\theta_1-\theta_2) + \bar{g_6}a_2^2a_1\sin(\theta_1-\theta_2)] - l_{10}[-\bar{f_1}a_2^2a_1$$
$$\sin(\theta_1-\theta_2) + \bar{f_2}a_1^3\sin(\theta_1-\theta_2) + \bar{f_3}a_1a_2^2\sin(\theta_1-\theta_2)$$
$$+ \bar{f_4}a_1^2a_2\sin 2(\theta_1-\theta_2)] \quad (3.51)$$

$$\dot{a_2}\dot{\theta_2} = a_2(\sigma_1-\sigma_2) + l_3a_1\sin(\theta_1-\theta_2) - l_4\cos(\theta_2) - l_1[l_2a_2 + l_3a_1\cos(\theta_1-\theta_2)$$
$$- l_4\sin(\theta_2)] - l_5[l_6a_1\cos(\theta_1-\theta_2) + l_7a_2 - l_8\sin(\theta_2)] - l_9[\bar{g_1}a_1^2a_2\cos 2(\theta_1-\theta_2)$$
$$+ \bar{g_2}a_2^3 + \bar{g_3}a_2a_1^2 + \bar{g_4}a_2^2a_1\cos(\theta_1-\theta_2) + \bar{g_5}a_1^3\cos(\theta_1-\theta_2)$$
$$+ \bar{g_6}a_2^2a_1\cos(\theta_1-\theta_2)] + l_{10}[\bar{f_1}a_2^2a_1\cos(\theta_1-\theta_2) + \bar{f_2}a_1^3\cos(\theta_1-\theta_2)$$
$$+ \bar{f_3}a_1a_2^2\cos(\theta_1-\theta_2) + \bar{f_4}a_1^2a_2\cos 2(\theta_1-\theta_2) + \bar{f_5}a_2^3 + \bar{f_6}a_2a_1^2] \quad (3.52)$$

where, $h_1,\ h_2, ...,\ h_{10}$ and $l_1,\ l_2, ...l_{10}$ are given in Appendix B.

To obtain the equilibrium solution, we set time derivatives terms to zero in Eqns. (3.49)–(3.52) and solve the resulting equations. Finally, the response of the beam upto second term can be written using Eqns. (3.1), (3.7), (3.23) and (3.44), and $\epsilon = 1$ as,

$$
\begin{aligned}
P_1(t) = x_{11} + x_{12} = \left(1 + i\frac{\Lambda_{11}}{2}\right) a_1 \cos(\omega_1 t + \beta_1) + \left(1 + i\frac{\Lambda_{12}}{2}\right) a_2 \cos(\omega_2 t + \beta_2) \\
+ \Lambda_{13} \cos[(\omega_1 + \sigma_1)t] + \frac{1}{2} c_{11} a_1^2 \cos 2(\omega_1 t + \beta_1) + \frac{1}{2} c_{12} a_1^2 + \frac{1}{2} c_{13} a_2^2 \cos 2(\omega_2 t + \beta_2) \\
+ \frac{1}{2} c_{14} a_2^2 + \frac{1}{2} c_{15} a_1 a_2 \cos 2[(\omega_1 - \omega_2)t + \beta_1 - \beta_2] \\
+ \frac{1}{2} c_{16} a_1 a_2 \cos 2[(\omega_1 + \omega_2)t + \beta_1 + \beta_2] \quad (3.53)
\end{aligned}
$$

$$
\begin{aligned}
P_2(t) = x_{21} + x_{22} = k_1 \left(1 + i\frac{\Lambda_{11}}{2}\right) a_1 \cos(\omega_1 t + \beta_1) + k_2 \left(1 + i\frac{\Lambda_{12}}{2}\right) a_2 \cos(\omega_2 t \\
+ \beta_2) + \Lambda_{14} \cos[(\omega_1 + \sigma_1)t] + \frac{1}{2} c_{21} a_1^2 \cos 2(\omega_1 t + \beta_1) + \frac{1}{2} c_{22} a_1^2 + \frac{1}{2} c_{23} a_2^2 \\
\cos 2(\omega_2 t + \beta_2) + \frac{1}{2} c_{24} a_2^2 + \frac{1}{2} c_{25} a_1 a_2 \cos 2[(\omega_1 - \omega_2)t + \beta_1 - \beta_2] \\
+ \frac{1}{2} c_{26} a_1 a_2 \cos 2[(\omega_1 + \omega_2)t + \beta_1 + \beta_2] \quad (3.54)
\end{aligned}
$$

where, the terms $\Lambda_{11},\ \Lambda_{12},\ \Lambda_{13}$ and Λ_{14} are defined as,

$$
\begin{aligned}
\Lambda_{11} &= \left[\frac{B_1}{\omega_1 B_4} + \frac{B_6}{\omega_2 B_8}\right],\ \Lambda_{12} = \left[\frac{B_2}{\omega_1 B_4} + \frac{B_5}{\omega_2 B_8}\right], \\
\Lambda_{13} &= \left[\frac{B_3}{\omega_1 B_4} + \frac{B_7}{\omega_2 B_8}\right],\ \Lambda_{14} = \left[\left[k_1 \frac{B_3}{\omega_1 B_4} + k_2 \frac{B_7}{\omega_2 B_8}\right]\right]. \quad (3.55)
\end{aligned}
$$

4 Results and discussion

In this section, we first study the linear frequency variation of in-plane and out-of-plane modes of a microbeam to locate the coupling region. Subsequently, we validate the modulation equations developed by the method of multiple scales with numerical solution obtained by solving the modal dynamic equations. Finally, we use the method of multiple scale to study coupled nonlinear response near and away from the coupling region. Additionally, we also analyze the influence of quality factor on the non-linear frequency response near the coupled region. To do the study, we consider the dimensions, material properties and electrostatic force coefficients in a fixed-fixed microbeam as mentioned in [2] and are given in Table 1.

4.1 Linear frequency analysis

To analyze the variation of linear frequency of two transverse modes of a fixed-fixed microbeam, we numerically solve the nonlinear static equations (given in Appendix

Table 1 Dimensions, material properties and the electrostatic force coefficients in a fixed-fixed microbeam [2]

Quantity	Symbol	Fixed-fixed beam
Length	L	500 μm
Width	B	4 μm
Height	H	200 nm
Side gap	g_0, g_1	4.5 μm, 7 μm
Bottom gap	d	500 μm
Youngs modulus	E	2.58×10^{10} N/m^2
Initial tension	N_0	38.336 μN
Density	ρ	3227.4 kg/m^3
Electric constant	ϵ_0	8.854×10^{-12} F/m
Fringing coefficients	k_1, k_2, k_3	0.945, 2.6, 1.3

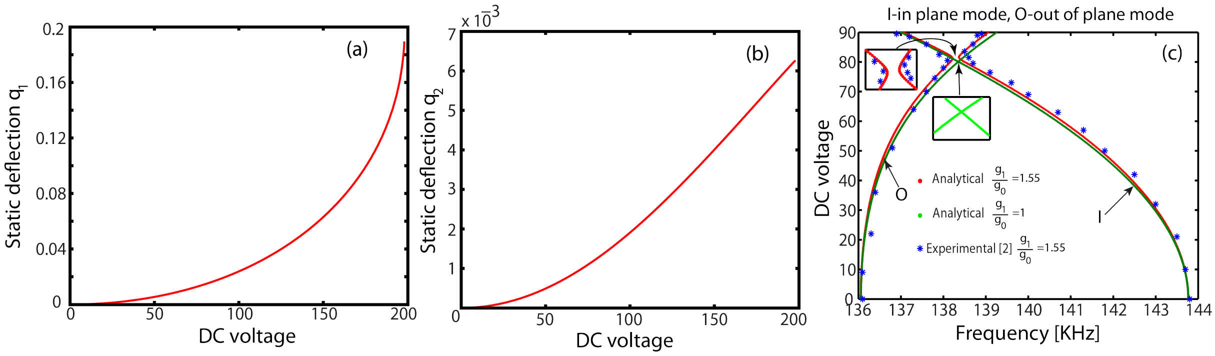

Fig. 2 (a) Variation of static deflection q_1 verses DC voltage for in-plane mode. (b) Variation of static deflection q_2 verses DC voltage for out-of-plane mode.(c) Variation of in-plane and out-of-plane frequencies with DC voltage for equal interbeam gaps (($g_0 = g_0 = 4.5\mu m$) and unequal interbeam gaps ($g_0 = 4.5\mu m$, $g_1 = 7\ \mu m$) and their comparison with experimental results [2]. It also shows 1:1 internal resonance at $V_{dc} = 81$ V.

A) and linear modal dynamic equations given by Eqns. (2.19) and (2.20) as described in [2]. Subsequently, we obtain linear frequencies corresponding to both the modes from Eqn. (2.21). Figures 2 (a) and (b) show the variation of static deflection verses DC voltage for in-plane and out-plane modes with a pull-in voltage of about $V_{dc} = 199$ V. Figure 2(c) shows the variation of in-plane and out-of-plane linear frequencies verses DC voltage and their comparison with experiments from [2]. As the DC voltage is varied from 0 to 90 V, the in-plane frequency decreases due to electrostatic softening effect and the out-of-plane frequency increases due to stretching of the beam in the in-plane direction. Consequently, the two frequencies come near to each other and, they, eventually, show 1:1 internal resonance at DC voltage of 81 V. We define this point as coupling point or region. It also shows the variation in-plane and out-of-plane frequencies with DC voltage for equal interbeam gaps ($g_0 = g_0 = 4.5\mu m$) with no coupling. In the following section, we apply the method of multiple scales to find nonlinear frequency response near the coupling region. We also compare the nonlinear response of different modes when the operating linear frequencies are below the coupling range.

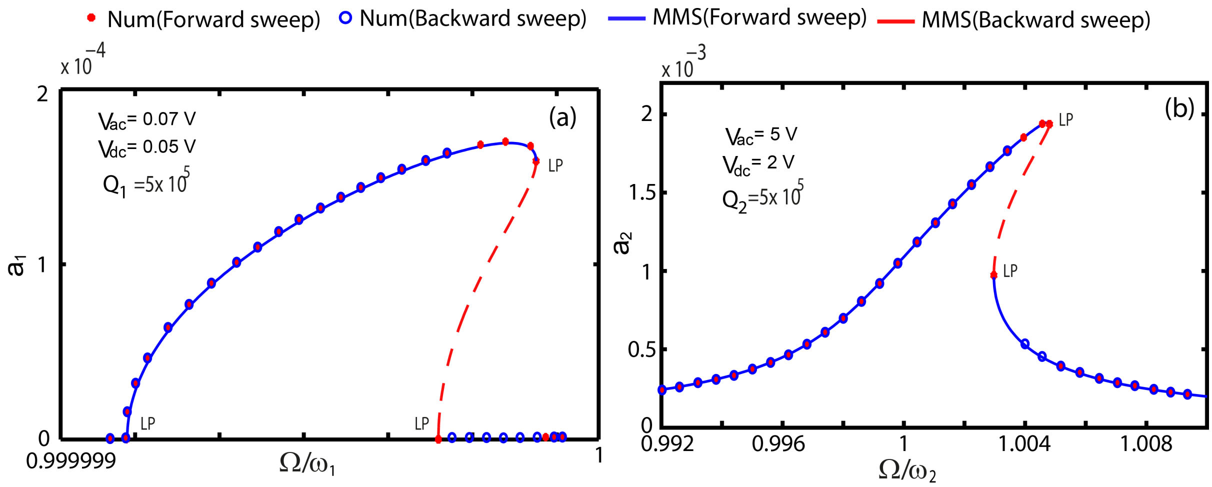

Fig. 3 (a) Uncoupled frequency verses $\frac{\Omega}{\omega_1}$ showing parametric response for in-plane mode. (b) Uncoupled nonlinear frequency verses $\frac{\Omega}{\omega_2}$ showing duffing response for out-of-plane mode. Here, LP denotes limit point.

4.2 Nonlinear response of uncoupled in-plane and out-of-plane modes

At very low DC voltage, the linear frequencies of in-plane and out-of-plane modes do not show any coupling region. To find the nonlinear response of uncoupled modes much below the coupling region, we neglect the coupling terms from Eqns. (2.17) and (2.18) and solve the governing equations of each modes, separately. To solve the non-linear dynamic equation, we consider only the dynamic component as the static deflection is negligible at low DC voltage. Subsequently, the method of multiple scales can be used to obtain the modulation equations for each modes, separately. While, the nonlinear response of in-plane mode turns out to be purely parametric, the nonlinear response of out-of-plane mode shows Duffing like response under the influence of direct and parametric forces [15]. Figure 3 shows uncoupled nonlinear frequency verses $\frac{\Omega}{\omega_1}$ showing parametric response for in-plane mode in Fig. 3(a) when $V_{ac} = 0.07$ V, $V_{dc} = 0.05$ V and $Q_1 = 5 \times 10^5$. Figure 3(b) shows uncoupled nonlinear response verses $\frac{\Omega}{\omega_2}$ for the out-of-plane mode when $V_{ac} = 5$ V, $V_{dc} = 2$ V and $Q_1 = 5 \times 10^5$. The values of AC and DC voltages are selected to show bi-stability region in the nonlinear response. Now, we present coupled nonlinear frequency response of two modes near the coupling region using the methods of multiple scales presented in the theoretical section.

4.3 Validation of MMS solution near coupling region

To validate the solutions obtained by solving the modulation Eqns. (3.49), (3.50), (3.51), and (3.52) from the method of multiple scales (MMS) near the coupling region, we solve the original modal dynamic Eqns. (2.17) and (2.18) using the Runge-Kutte method. To compare the results, we convert a_1 and a_2 appearing in the modulation equations to equivalent expression of $P_1(t)$ and $P_2(t)$ as given by

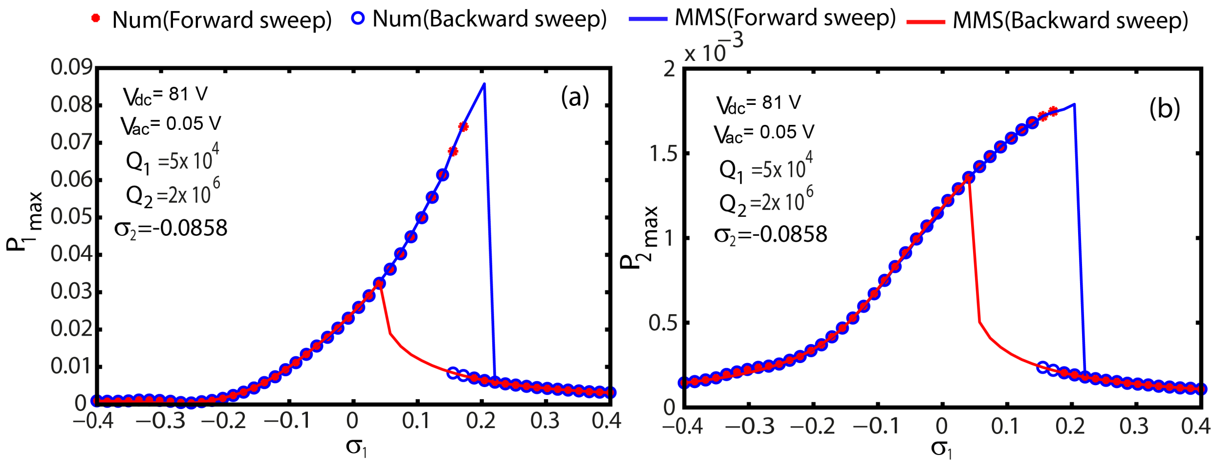

Fig. 4 (a) Comparison of numerical results with solutions based on MMS for in-plane mode at coupling point (b) Comparison of numerical results with solutions based on MMS for out-of-plane mode at coupling point . Here, we take $V_{dc} = 81$ V, $V_{ac} = 0.05$ V, $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$, $\sigma_2 = -0.0858$, respectively.

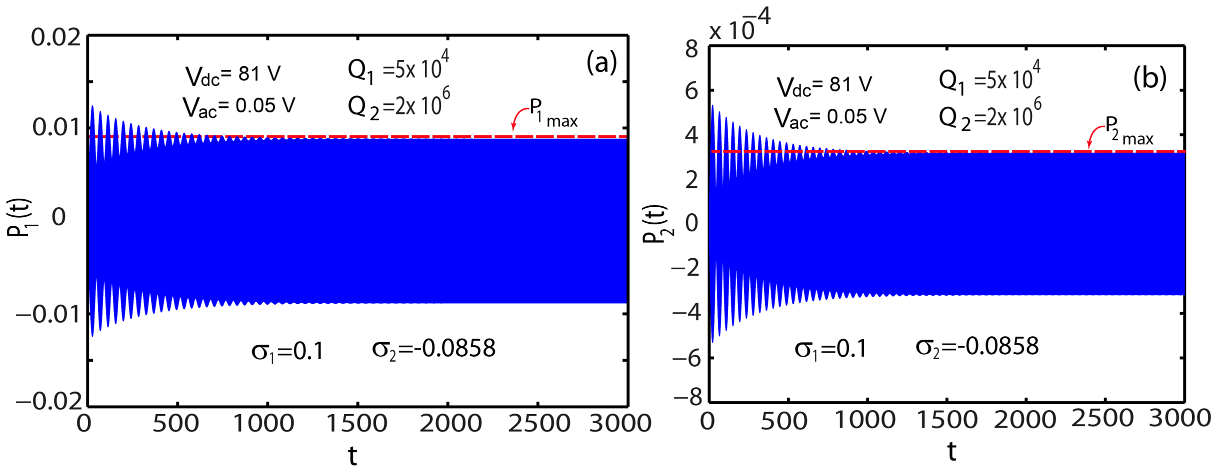

Fig. 5 Long-time histories at coupling point for $V_{dc} = 81$ V, $V_{ac} = 0.05$ V, $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$, when $\sigma_1 = 0.1$ and $\sigma_2 = -0.0858$.

Eqns. (3.53) and (3.54). Thus, $P_1(t)$ and $P_2(t)$ obtained from MMS are compared with the solutions obtained from the original equations. Figures 4(a) and (b) show comparisons between numerical results and the solutions based on MMS for in-plane and out-of-plane modes near the coupling point for the parameter values $V_{dc} = 81$ V, $V_{ac} = 0.05$ V, $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$ and $\sigma_2 = -0.0858$. Figures 5(a) and (b) show the long-time histories of the response for in-plane and out-of-plane modes when $\sigma_1 = 0.1$ and $\sigma_2 = -0.0.0858$. The time histories show that the steady state response of in-plane and out-of-plane modes consists of single frequency.

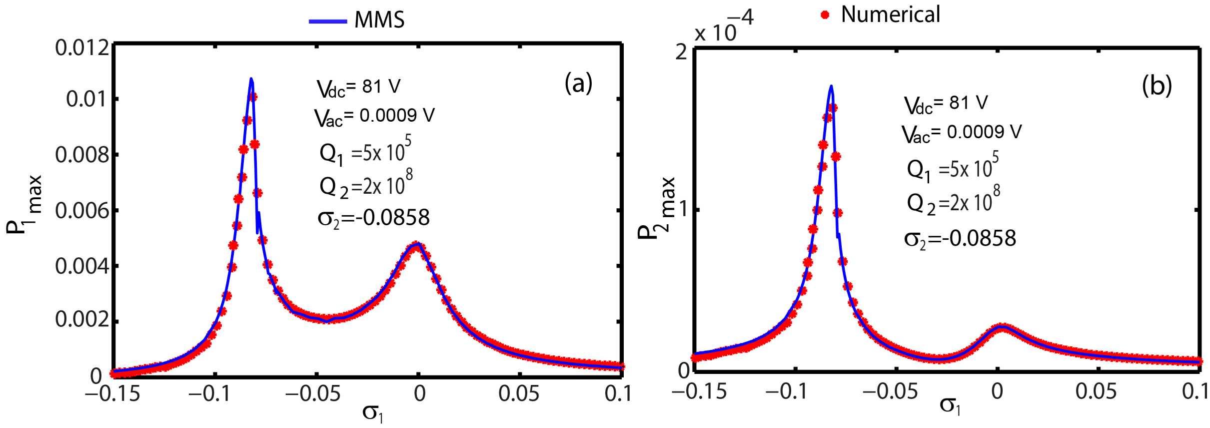

Fig. 6 (a) Comparison of numerical results with solutions based on MMS for in-plane mode at coupling point (b) Comparison of numerical results with solutions based on MMS for out-of-plane mode at coupling point . Here, we take $V_{dc} = 81$ V, $V_{ac} = 0.0009$ V, $Q_1 = 5 \times 10^5$, $Q_2 = 2 \times 10^8$, $\sigma_2 = -0.0858$, respectively.

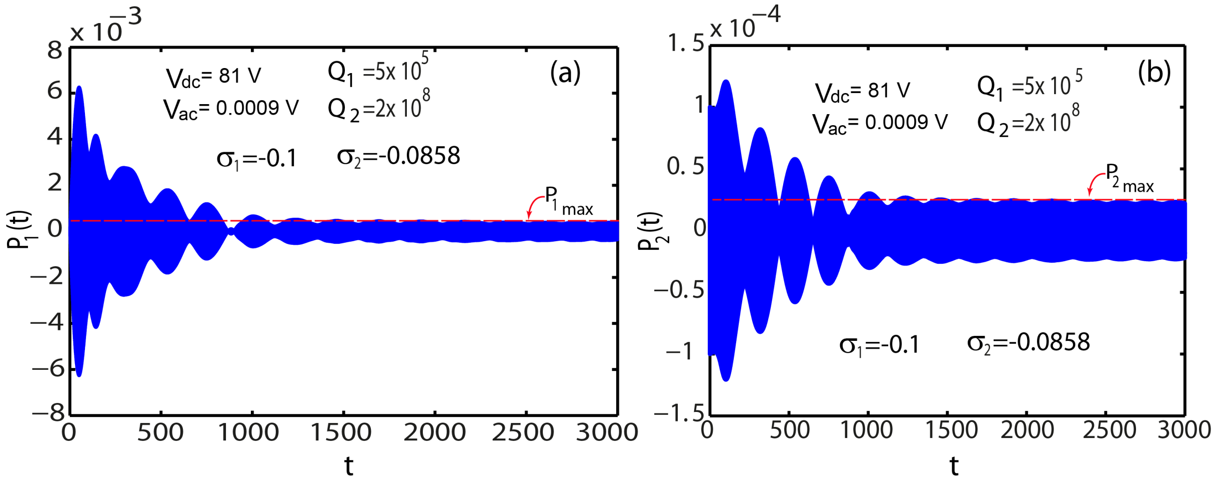

Fig. 7 Long-time histories at coupling point for $V_{dc} = 81$ V, $V_{ac} = 0.0009$ V, $Q_1 = 5 \times 10^5$, $Q_2 = 2 \times 10^8$, when $\sigma_1 = -0.1$ and $\sigma_2 = -0.0858$.

Similarly, the comparisons between the numerical results and the solutions based on MMS for in-plane and out-of-plane modes at coupling point ($V_{dc} = 81$ V and $V_{ac} = 0.0009$ V) at different quality factors $Q_1 = 5 \times 10^5$ and $Q_2 = 2 \times 10^8$ are shown in Figs. 6(a) and (b). In this case, both in-plane and out-of-plane frequency response show two peaks. The long-time histories as shown in Figs. 7(a) and (b) for $\sigma_1 = -0.1$ and $\sigma_2 = -0.0858$ also show that the steady state response of both in-plane and out-of-plane modes consist of two frequencies corresponding to two peaks appearing in the response. Thus, it shows clearly the influence of one mode on another.

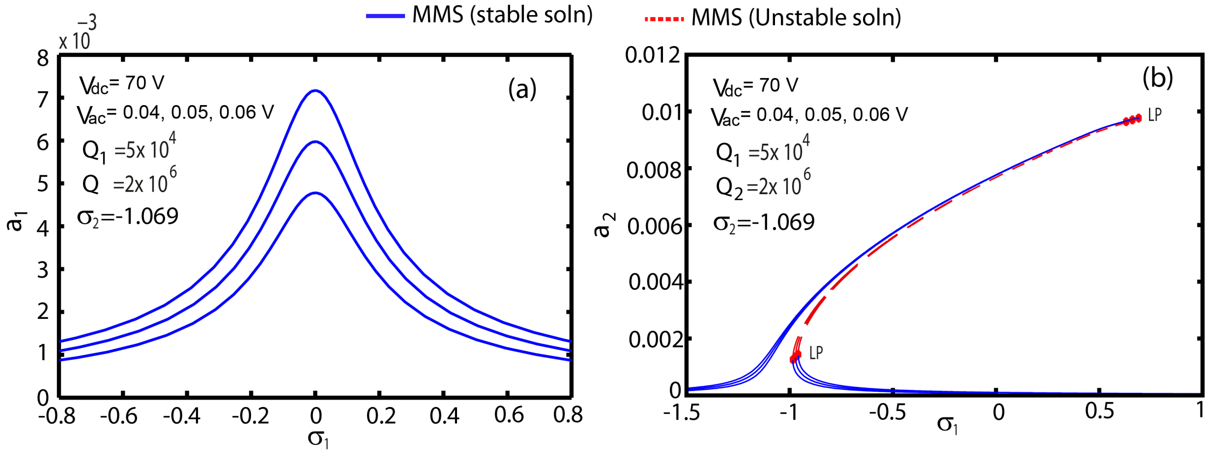

Fig. 8 (a) Frequency response for different AC voltages below coupling point corresponding to (a) in-plane mode (b) out-of-plane mode. Here, $V_{dc} = 70$ V, $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$, $\sigma_2 = -1.069$. LP denotes limit point.

4.4 Nonlinear response near and below the coupling region

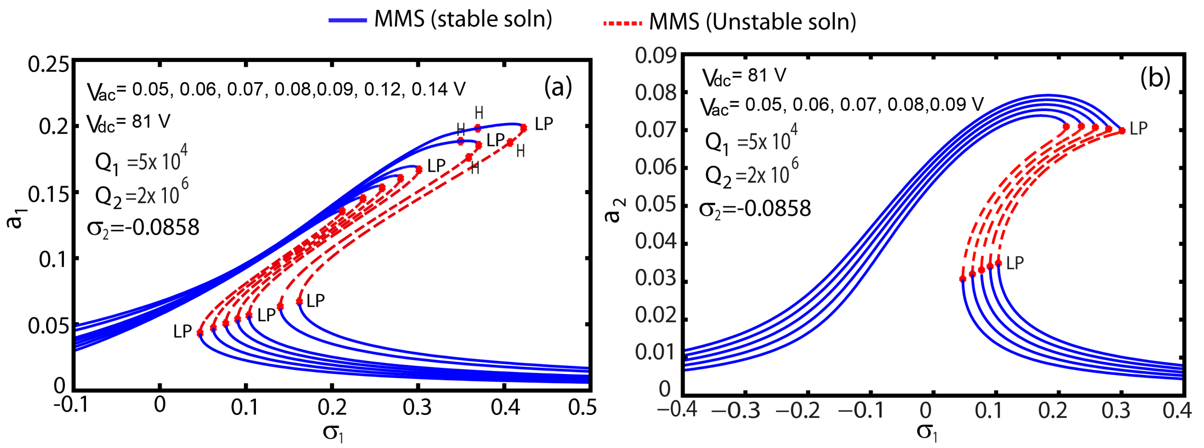

Fig. 9 Frequency response for different AC voltages at coupling point corresponding to (a) in-plane mode (b) out-of-plane mode. Here, $V_{dc} = 81$ V, $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$, $\sigma_2 = -0.0858$. LP denotes limit point and H denotes hopf bifurcation point.

In this section, to show the influence of coupling on nonlinear response near and below the coupling region, we analyze the variation of a_1 and a_2 corresponding to in-plane and out-of-plane modes. For the linear frequency relation below the coupling region at $V_{dc} = 70$ V and near the coupling region at $V_{dc} = 81$ V, we take $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$. Figures 8(a) and (b) show the frequency response for different AC voltages below coupling point along in-plane and out-of-plane directions. With the increase in V_{ac} from 0.04 to 0.06, the response amplitude increases

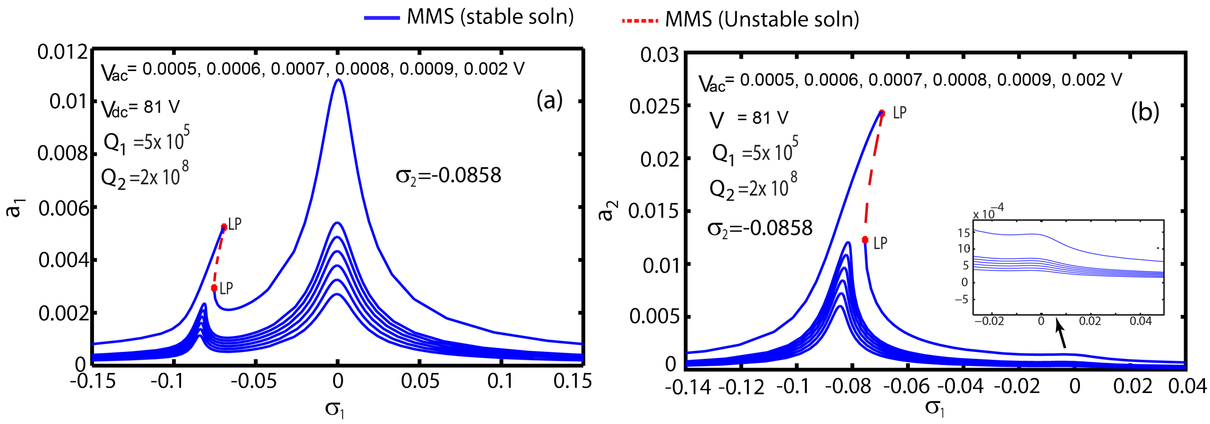

Fig. 10 Frequency response for different AC voltages at coupling point corresponding to (a) in-plane mode (b) out-of-plane mode. Here, $V_{dc} = 81$ V, $Q_1 = 5 \times 10^5$, $Q_2 = 2 \times 10^8$, $\sigma_2 = -0.0858$. LP denotes limit point.

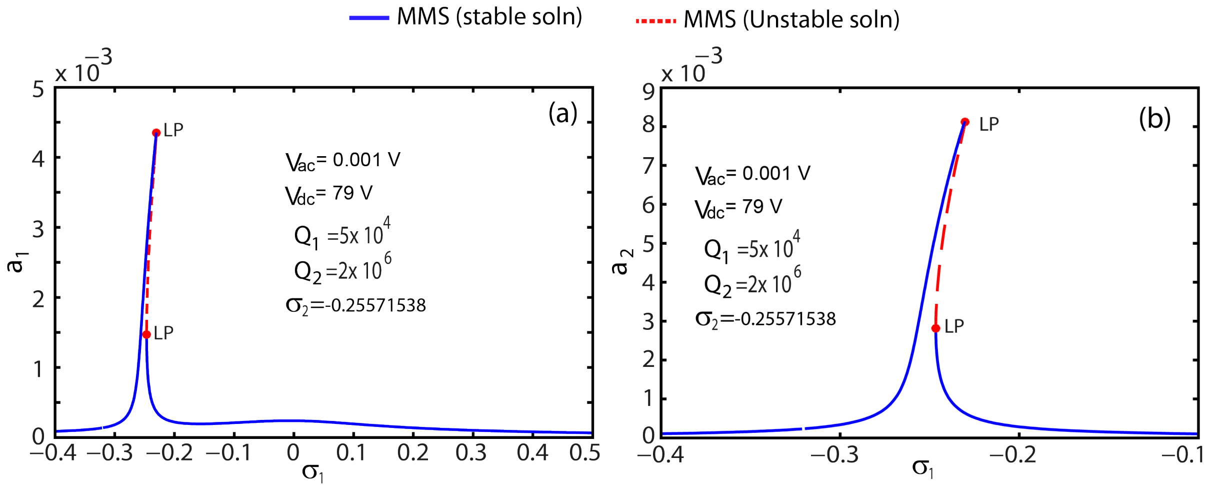

Fig. 11 Frequency response below coupling point corresponding to (a) in-plane mode (b) out-of-plane mode. Here, $V_{dc} = 79$ V, $V_{ac} = 0.001$ V, $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$, $\sigma_2 = -0.25571538$. LP denotes limit point.

in both the cases. However, only single peak is observed in both the cases due to relatively low quality factor. The frequency response for in-plane motion is found to be linear, whereas, out-of-plane motion shows nonlinear response. By operating the beam near the coupling region at $V_{dc} = 81$ V, the nonlinear coupled response of two modes shows combined effect of parametric and Duffing like response when the quality factors remains same as $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$ which are shown in Figures 9 (a) and (b). The coupling between parametric and duffing response at coupling point is due to simultaneous parametric and direct excitation of microbeam by two symmetrically placed side electrodes and a bottom electrode [15]. It is also observed that as AC voltage V_{ac} is increased from 0.05 to 0.14 V, the

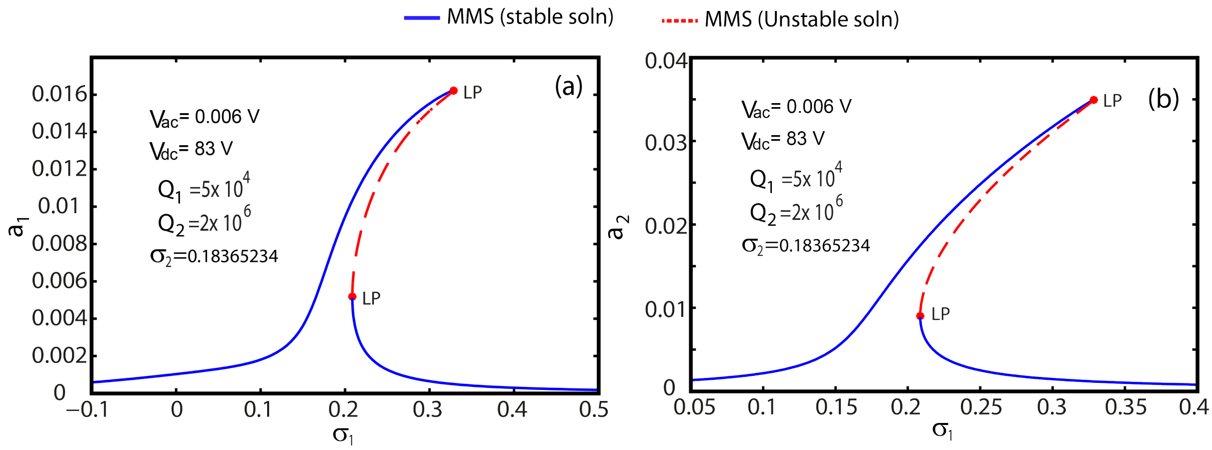

Fig. 12 Frequency response above coupling point corresponding to (a) in-plane mode (b) out-of-plane mode. Here, $V_{dc} = 83$ V, $V_{ac} = 0.06$ V, $Q_1 = 5 \times 10^4$, $Q_2 = 2 \times 10^6$, $\sigma_2 = 0.18365234$. LP denotes limit point.

response amplitude and the bandwidth gradually increase with increasing hardening effect. To see the influence of quality factor on the non-linear coupled response near the coupling region, we take another set of quality factors $Q_1 = 5 \times 10^5$ and $Q_2 = 2 \times 10^8$ as shown in Figs. 10 (a) and (b). It is observed that coupled response shows two peaks, thus, clearly indicating the influence of one mode on another. With further increase in AC voltage, V_{ac}, from 0.0005 to 0.002 V, the response amplitude of two modes increase gradually along both directions and frequency response of one of the two modes becomes nonlinear showing hardening effect when V_{ac} is more than 0.0009 V. Figures 11(a) and (b) show the frequency response of the in-plane and out-of-plane directions below the coupling point at DC voltage $V_{dc} = 79$ V when $V_{ac} = 0.001$ V, $Q_1 = 5 \times 10^4$ and $Q_2 = 2 \times 10^6$. Similarly, the frequency response along the in-plane and out-of-plane directions above the coupling point at a DC voltage of $V_{dc} = 83$ V for $V_{ac} = 0.006$ V and same values of quality factors are shown in Figures 12(a) and (b). The results in both the cases show that the coupled effect reduces drastically as we go up or below the coupled region.

Finally, we state that the tuning of nonlinear frequency response of two modes near and below the coupling region by the application DC voltage and quality factors. The study presented in this paper can also be extended to understand the coupling of different modes of beams in MEMS arrays.

5 Conclusion

In this paper, we have developed a theoretical model for in-plane and out-of-plane motions of a fixed-fixed microbeam separated from two symmetrically placed side electrodes and a bottom electrode. Using the electrostatic force model based on the direct and fringing forces, we obtain the partial differential equations governing the nonlinear motion of in-plane and out-of-plane motions. To do linear and nonlinear analysis, we obtain the reduced order form of the equations using the Galerkin's method. To analyze the variation of two modes at different DC voltage, we plot linear frequencies versus DC voltage. We found that the two modes shows coupling at around DC voltage of 81 V. Thus, we obtain 1:1 internal resonance condition near the coupling region. To find the nonlinear response near and below the coupling region, we apply the method of multiple scales (MMS). After validating the solution from MMS with numerical solution near the coupling region, we analyze the influence of ac voltage and quality factor on the nonlinear response at and near the coupling point. We found that the nonlinear response below the coupling point shows uncoupled response of each modes, the response near the coupling region shows different types of coupled response at different quality factors.

Acknowledgement

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Appendix A

Nonlinear static equations

$$\begin{aligned}
&758.48\, {q_1}^7\alpha_1 + (1022.84\, r_1\alpha_1 - 1022.84\, \alpha_1)\, {q_1}^6 + \big(347.07\, \alpha_1 + 347.07\, {r_1}^2\alpha_1 + \\
&758.48\alpha_2 {q_2}^2 + 61.66\, N + 1934.02 - 1388.29\, r_1\alpha_1\big){q_1}^5 + \big(2654.13\, r_1 - 470.92 \\
&{r_1}^2\alpha_1 - 2654.13 - 1022.84\alpha_2 {q_2}^2 + 83.15\, r_1 N + 1022.84\, r_1\alpha_2 {q_2}^2 + 470.92\, r_1\alpha_1 \\
&-83.15\, N\big){q_1}^4 + \big(347.07 {r_1}^2\alpha_2 {q_2}^2 + 927.00 + 28.21\, N - 3708.01\, r_1 + 927.00\, {r_1}^2 \\
&+151.34\, {r_1}^2\alpha_1 - 112.85\, r_1 N - 1388.29 r_1\alpha_2 {q_2}^2 + 347.07\, \alpha_2 {q_2}^2 + 28.21\, {r_1}^2 N\big){q_1}^3 \\
&+\big(38.28\, r_1 N - 1.33\, \beta_s {V_{10}}^2 + 1330.88\, r_1 - 470.92\, {r_1}^2\alpha_2 {q_2}^2 + 470.92\, r_1\alpha_2 {q_2}^2 \\
&-1330.88\, {r_1}^2 - 38.28\, {r_1}^2 N + 1.33\, \beta_s {V_{12}}^2\big){q_1}^2 + \big(12.30\, {r_1}^2 N - 2.00\, \beta_s {V_{12}}^2 \\
&-2.00\, \beta_s {V_{10}}^2 r_1 + 500.56\, {r_1}^2 + 151.34\, {r_1}^2\alpha_2 {q_2}^2\big) q_1 + 0.83\, \beta_s {V_{12}}^2 \\
&-0.83\, \beta_s {V_{10}}^2 {r_1}^2 = 0
\end{aligned}$$

$$\begin{aligned}
&347.07\, {q_2}^5\alpha_3\alpha_2 - 470.92\, {q_2}^4\alpha_3\alpha_2 + \big(151.34\, \alpha_3\alpha_2 - 1.85\, {V_g}^2\beta_{3\,g} + 927.0 + \\
&347.07\alpha_3\alpha_1 {q_1}^2 + 1.85\, {V_{10}}^2\alpha_{2\,g} + 1.85\, {V_{12}}^2\alpha_{2\,g} + 28.21\, \alpha_3 N\big){q_2}^3 + \big(- 2.66 \\
&{V_{12}}^2\alpha_{2\,g} - 38.28\, \alpha_3 N - 2.66\, {V_{10}}^2\alpha_{2\,g} - 1330.88 + 1.33\, {V_{10}}^2\alpha_g + 1.33\, {V_{12}}^2\alpha_g \\
&+3.99 {V_g}^2\beta_{3\,g} - 1.33\, {V_g}^2\beta_{2\,g} - 470.92\, \alpha_3\alpha_1 {q_1}^2\big){q_2}^2 + \big(- 2.0\, {V_{12}}^2\alpha_g + 500.56 \\
&+151.34\alpha_3\alpha_1 {q_1}^2 - 3.0\, {V_g}^2\beta_{3\,g} + 2.0\, {V_g}^2\beta_{2\,g} + 12.30\, \alpha_3 N + 1.0\, {V_{12}}^2\alpha_{2\,g} \\
&+1.0\, {V_{10}}^2\alpha_{2\,g} - 2.0\, {V_{10}}^2\alpha_g\big) q_2 - 0.831\, {V_g}^2\beta_g - 0.831\, {V_g}^2\beta_{2\,g} \\
&+0.831\, {V_g}^2\beta_{3\,g} + 0.831\, {V_{10}}^2\alpha_g + 0.831\, {V_{12}}^2\alpha_g = 0
\end{aligned}$$

In-plane equation coefficients

$$\begin{aligned}
m_{1i} = \;& 5.56\, {q_1}^2{r_1}^3 - 11.59\, {q_1}^4{r_1}^2 - 17.11\, {q_1}^5 r_1 + 3.98\, {r_1}^2 q_1 - 2.65\, {q_1}^3{r_1}^3 \\
&+ 23.86\, {r_1}^2{q_1}^3 - 3.98\, q_1\, {r_1}^3 + 34.77\, {q_1}^4 r_1 - 16.67\, {q_1}^2{r_1}^2 - 23.86\, {q_1}^3 r_1 \\
&+ 5.55\, r_1 {q_1}^2 + {r_1}^3 + 2.65\, {q_1}^3 - 8.49\, {q_1}^6 - 11.59\, {q_1}^4 + 17.11\, {q_1}^5
\end{aligned}$$

$$
\begin{aligned}
k_{1i} =& - 511.42\, q_1{}^3 r_1{}^3 \alpha_2 q_2{}^2 + 2119.14\, r_1{}^2 q_1{}^3 \alpha_1 - 1534.26 q_1{}^5 r_1{}^3 \alpha_1 - 11.11 \beta_s V_{12}{}^2 \\
& q_1{}^2 + 151.34\, r_1{}^3 \alpha_2 q_2{}^2 + 276.20\, q_1{}^5 N - 6826.314\, q_1{}^6 r_1{}^2 \alpha_1 + 13808.31\, q_1{}^5 \\
& r_1{}^2 \alpha_1 - 41.57\, q_1{}^3 r_1{}^3 N - 2\, \beta_s V_{12}{}^2 - 11.11\, \beta_s V_{10}{}^2 r_1 q_1{}^2 + 554.90\, q_1{}^4 r_1 N \\
& + 41.57\, q_1{}^3 N + 374.15\, r_1{}^2 q_1{}^3 N + 500.56\, r_1{}^3 - 5802.07\, q_1{}^4 + 8563.10\, q_1{}^5 \\
& - 253.92\, q_1{}^2 r_1{}^2 N + 1327.064733\, q_1{}^3 - 4252.39 q_1{}^6 + 2781.01 q_1{}^2 r_1{}^3 + 57.42 \\
& r_1{}^2 q_1\, N + 511.42\, q_1{}^3 \alpha_2 q_2{}^2 - 5802.07\, q_1{}^4 r_1{}^2 - 8563.10\, q_1{}^5 r_1 + 1996.33 r_1{}^2 \\
& q_1 - 1327.06\, q_1{}^3 r_1{}^3 + 11943.58\, r_1{}^2 q_1{}^3 - 1996.33\, q_1\, r_1{}^3 + 17406.21\, q_1{}^4 r_1 \\
& - 8343.02\, q_1{}^2 r_1{}^2 - 11943.58\, q_1{}^3 r_1 + 2781.01\, r_1 q_1{}^2 + 3397.83\, q_1{}^5 \alpha_2 q_2{}^2 - \\
& 3123.66\, q_1{}^2 r_1{}^2 \alpha_2 q_2{}^2 + 6826.31\, q_1{}^4 r_1 \alpha_2 q_2{}^2 - 184.97\, q_1{}^4 N - 138.32\, q_1{}^6 N \\
& + 12.31\, r_1{}^3 N - 6826.31\, q_1{}^6 \alpha_1 + 10193.49\, q_1{}^7 \alpha_1 - 5104.69\, q_1{}^8 \alpha_1 + 1534.26 \\
& q_1{}^5 \alpha_1 + 454.01 q_1{}^2 r_1{}^3 \alpha_1 - 13808.31 q_1{}^5 r_1 \alpha_1 - 10193.49 q_1{}^7 r_1 \alpha_1 + 3123.66 \\
& q_1{}^4 r_1{}^3 \alpha_1 + 7.98\, \beta_s V_{12}{}^2 q_1 + 84.64\, q_1{}^2 r_1{}^3 N + 3123.66 q_1{}^4 r_1 \alpha_1 - 276.21\, q_1{}^5 \\
& r_1 N - 2119.14\, q_1{}^3 r_1{}^3 \alpha_1 + 5.30\, \beta_s V_{12}{}^2 q_1{}^3 + 20478.94\, q_1{}^6 r_1 \alpha_1 - 5.30\, \beta_s \\
& V_{10}{}^2 q_1{}^3 - 374.15\, q_1{}^3 r_1 N - 57.42\, q_1\, r_1{}^3 N - 2\, \beta_s V_{10}{}^2 r_1{}^3 - 9370.99\, q_1{}^4 r_1{}^2 \\
& \alpha_1 - 2275.44\, q_1{}^4 \alpha_2 q_2{}^2 - 1701.56\, q_1{}^6 \alpha_2 q_2{}^2 + 84.64\, r_1 q_1{}^2 N + 706.38\, r_1{}^2 \\
& q_1\, \alpha_2 q_2{}^2 - 7.97\, \beta_s V_{10}{}^2 r_1{}^2 q_1 - 706.38\, q_1\, r_1{}^3 \alpha_2 q_2{}^2 - 4602.77\, q_1{}^3 r_1 \alpha_2 q_2{}^2 \\
& + 1041.22\, q_1{}^2 r_1{}^3 \alpha_2 q_2{}^2 + 4602.77\, r_1{}^2 q_1{}^3 \alpha_2 q_2{}^2 - 3397.83\, q_1{}^5 r_1 \alpha_2 q_2{}^2 \\
& - 184.97\, q_1{}^4 r_1{}^2 N - 2275.44\, q_1{}^4 r_1{}^2 \alpha_2 q_2{}^2 + 1041.22\, r_1 q_1{}^2 \alpha_2 q_2{}^2
\end{aligned}
$$

$$
\begin{aligned}
n_{1i} =& \quad 1041.22\, q_1{}^2 r_1{}^3 \alpha_1 + 511.42\, q_1{}^3 \alpha_1 - 706.38\, q_1\, r_1{}^3 \alpha_1 + 151.33\, r_1{}^3 \alpha_1 - \\
& 3123.66\, r_1{}^2 q_1{}^2 \alpha_1 + 706.38\, r_1{}^2 q_1\, \alpha_1 + 6826.31\, q_1{}^4 r_1 \alpha_1 - 2275.44\, q_1{}^4 \alpha_1 \\
& - 511.42\, q_1{}^3 r_1{}^3 \alpha_1 - 2275.44\, q_1{}^4 r_1{}^2 \alpha_1 - 1701.56\, q_1{}^6 \alpha_1 + 1041.22\, r_1 q_1{}^2 \alpha_1 \\
& + 74602.77 r_1{}^2 q_1{}^3 \alpha_1 - 3397.83 q_1{}^5 r_1 \alpha_1 + 3397.83 q_1{}^5 \alpha_1 - 4602.77 q_1{}^3 r_1 \alpha_1
\end{aligned}
$$

$$
\begin{aligned}
n_{2i} =& - 6826.31 q_1{}^5 r_1{}^2 \alpha_1 - 5104.69 q_1{}^7 \alpha_1 - 13808.31 q_1{}^4 r_1 \alpha_1 - 9370.99 r_1{}^2 q_1{}^3 \alpha_1 \\
& + 1534.26\, q_1{}^4 \alpha_1 + 10193.49\, q_1{}^6 \alpha_1 + 13808.31\, q_1{}^4 r_1{}^2 \alpha_1 + 454.01\, q_1\, r_1{}^3 \alpha_1 \\
& - 1534.26\, q_1{}^4 r_1{}^3 \alpha_1 + 3123.66\, q_1{}^3 r_1{}^3 \alpha_1 - 6826.31\, q_1{}^5 \alpha_1 - 2119.14\, q_1{}^2 r_1{}^3 \\
& \alpha_1 + 3123.66\, q_1{}^3 r_1 \alpha_1 + 20478.94\, q_1{}^5 r_1 \alpha_1 - 10193.49\, q_1{}^6 r_1 \alpha_1 \\
& + 2119.14\, r_1{}^2 q_1{}^2 \alpha_1
\end{aligned}
$$

$$
\begin{aligned}
n_{3i} = & - 4602.77\, q_1{}^3 r_1 \alpha_2 - 2275.44\, q_1{}^4 r_1{}^2 \alpha_2 - 3123.66\, q_1{}^2 r_1{}^2 \alpha_2 + 4602.77\, r_1{}^2 \\
& q_1{}^3 \alpha_2 - 3397.83 q_1{}^5 r_1 \alpha_2 - 1701.56 q_1{}^6 \alpha_2 + 706.38 r_1{}^2 q_1 \alpha_2 + 511.42 q_1{}^3 \\
& \alpha_2 + 151.34\, r_1{}^3 \alpha_2 + 6826.31 q_1{}^4 r_1 \alpha_2 + 1041.22 q_1{}^2 r_1{}^3 \alpha_2 + 1041.22 r_1 q_1{}^2 \\
& \alpha_2 - 511.42 q_1{}^3 r_1{}^3 \alpha_2 - 706.38 q_1 r_1{}^3 \alpha_2 - 2275.44 q_1{}^4 \alpha_2 + 3397.83 q_1{}^5 \alpha_2 \\
n_{4i} = & - 4550.87 q_1{}^4 \alpha_2 q_2 - 1022.84 q_1{}^3 r_1{}^3 \alpha_2 q_2 - 4550.87 q_1{}^4 r_1{}^2 \alpha_2 q_2 + 13652.63 \\
& q_1{}^4 r_1 \alpha_2 q_2 + 9205.54\, q_1{}^3 r_1{}^2 \alpha_2 q_2 - 6795.66\, q_1{}^5 r_1 \alpha_2 q_2 - 9205.54\, q_1{}^3 r_1 \\
& \alpha_2 q_2 + 1412.76\, r_1{}^2 q_1\, \alpha_2 q_2 - 1412.76\, q_1\, r_1{}^3 \alpha_2 q_2 - 6247.33\, r_1{}^2 q_1{}^2 \alpha_2 q_2 \\
& + 2082.44\, r_1 q_1{}^2 \alpha_2 q_2 + 2082.44\, q_1{}^2 r_1{}^3 \alpha_2 q_2 + 1022.84\, q_1{}^3 \alpha_2 q_2 + 302.67 \\
& r_1{}^3 \alpha_2 q_2 + 6795.66\, q_1{}^5 \alpha_2 q_2 - 3403.13\, q_1{}^6 \alpha_2 q_2 \\
n_{5i} = & - 2.0\, \beta_s r_1{}^3 - 7.98\, \beta_s r_1{}^2 q_1 - 11.11\, \beta_s r_1 q_1{}^2 - 5.30\, \beta_s q_1{}^3 \\
n_{6i} = & \quad 3.70\, \beta_s q_1{}^3 - 1.66\, \beta_s + 6.0\, \beta_s q_1 - 7.98\, \beta_s q_1{}^2
\end{aligned}
$$

$$
\begin{aligned}
n_{7i} = & \quad 1041.22 q_1{}^3 r_1 \alpha_2 - 706.38 q_1{}^2 r_1{}^3 \alpha_2 - 2275.44 q_1{}^5 r_1{}^2 \alpha_2 + 151.33\, q_1 r_1{}^3 \alpha_2 \\
& - 3397.83\, q_1{}^6 r_1 \alpha_2 - 4602.77\, q_1{}^4 r_1 \alpha_2 + 3397.83\, q_1{}^6 \alpha_2 + 511.42\, q_1{}^4 \alpha_2 + \\
& 1041.22\, q_1{}^3 r_1{}^3 \alpha_2 - 1701.56\, q_1{}^7 \alpha_2 - 3123.66\, r_1{}^2 q_1{}^3 \alpha_2 + 4602.77\, q_1{}^4 r_1{}^2 \\
& \alpha_2 - 511.42 q_1{}^4 r_1{}^3 \alpha_2 - 2275.44 q_1{}^5 \alpha_2 + 6826.31 q_1{}^5 r_1 \alpha_2 + 706.38 q_1{}^2 r_1{}^2 \alpha_2 \\
n_{8i} = & - 6795.66 q_1{}^6 r_1 \alpha_2 q_2 - 9205.54 q_1{}^4 r_1 \alpha_2 q_2 - 1412.76 q_1{}^2 r_1{}^3 \alpha_2 q_2 - 6247.33 q_1{}^3 \\
& r_1{}^2 \alpha_2 q_2 - 4550.87 q_1{}^5 r_1{}^2 \alpha_2 q_2 + 2082.44 q_1{}^3 r_1{}^3 \alpha_2 q_2 + 1412.76 r_1{}^2 q_1{}^2 \alpha_2 q_2 \\
& + 302.67\, q_1\, r_1{}^3 \alpha_2 q_2 + 1022.84\, q_1{}^4 \alpha_2 q_2 - 1022.84\, q_1{}^4 r_1{}^3 \alpha_2 q_2 + 6795.66\, q_1{}^6 \\
& \alpha_2 q_2 - 4550.87\, q_1{}^5 \alpha_2 q_2 + 2082.44\, q_1{}^3 r_1 \alpha_2 q_2 + 13652.63\, q_1{}^5 r_1 \alpha_2 q_2 \\
& - 3403.13\, q_1{}^7 \alpha_2 q_2 + 9205.54\, q_1{}^4 r_1{}^2 \alpha_2 q_2
\end{aligned}
$$

$$
\begin{aligned}
n_{9i} = & - 3.98\, \beta_s r_1 q_1{}^2 + 2.65\, \beta_s q_1{}^4 + 3.98\, \beta_s r_1{}^2 q_1{}^2 + 5.55\, \beta_s r_1 q_1{}^3 + \beta_s r_1{}^3 q_1 \\
& - 1.85\, \beta_s q_1{}^3 - 0.83\, \beta_s r_1{}^3 - 3.0\, \beta_s r_1{}^2 q_1 \\
n_{10i} = & - 3.0\, \beta_s q_1{}^2 - 2.49\, \beta_s r_1 q_1 - 1.33\, \beta_s r_1 q_1{}^3 + \beta_s r_1 - 1.85\, \beta_s q_1{}^4 + 3.98\, \beta_s q_1{}^3 \\
& + 0.83\, \beta_s q_1 + 3.0\, \beta_s r_1 q_1{}^2 \\
n_{11i} = & - 8.49\, q_1{}^6 c_1 - 11.59\, q_1{}^4 r_1{}^2 c_1 + 3.98\, r_1{}^2 q_1\, c_1 + 5.55\, q_1{}^2 r_1{}^3 c_1 + 5.55\, r_1 q_1{}^2 c_1 \\
& - 3.98\, q_1\, r_1{}^3 c_1 - 23.86\, q_1{}^3 r_1 c_1 + 17.11\, q_1{}^5 c_1 + 23.86\, q_1{}^3 r_1{}^2 c_1 + 2.65\, q_1{}^3 c_1 \\
& - 16.67\, r_1{}^2 q_1{}^2 c_1 - 11.59\, q_1{}^4 c_1 - 17.11\, q_1{}^5 r_1 c_1 - 2.65\, q_1{}^3 r_1{}^3 c_1 + r_1{}^3 c_1 \\
& + 34.77\, q_1{}^4 r_1 c_1
\end{aligned}
$$

$$
\lambda_1 = \sqrt{\frac{k_{1i}}{m_{1i}}}, \quad t_1 = \frac{n_{1i}}{m_{1i}}, \quad t_2 = \frac{n_{2i}}{m_{1i}}, \quad t_3 = \frac{n_{3i}}{m_{1i}}, \quad t_4 = \frac{n_{4i}}{m_{1i}}, \quad t_5 = \frac{n_{5i}}{m_{1i}},
$$

$$
t_6 = \frac{n_{6i}}{m_{1i}}, \quad t_7 = \frac{n_{7i}}{m_{1i}}, \quad t_8 = \frac{n_{8i}}{m_{1i}}, \quad t_9 = \frac{n_{9i}}{m_{1i}}, \quad t_{10} = \frac{n_{10i}}{m_{1i}}, \quad t_{11} = \frac{n_{11i}}{m_{1i}}
$$

Out-of-plane equation coefficients

$$
\begin{aligned}
m_o =& -2.65\,q_2{}^3\alpha_3+\alpha_3+5.56\,\alpha_3 q_2{}^2-3.98\,q_2\,\alpha_3\\
k_o =& -1996.33\,q_2-1327.06\,q_2{}^3+2781.01\,q_2{}^2-1534.26\,q_2{}^5\alpha_3\alpha_2-2.65\,\alpha_{2\,g}q_2{}^3\\
&V_{12}{}^2-41.57\,q_2{}^3\alpha_3 N+3.98\,V_{1g}{}^2\beta_{3\,g}q_2+5.56\,\alpha_{2\,g}q_2{}^2V_{12}{}^2+5.56\,\alpha_{2\,g}q_2{}^2\\
&V_{10}{}^2-3.98\,\alpha_{2\,g}q_2\,V_{10}{}^2+84.64\,\alpha_3 q_2{}^2N+3123.66\,q_2{}^4\alpha_3\alpha_2+151.34\,\alpha_3\\
&\alpha_1 q_1{}^2-5.56\,V_{1g}{}^2\beta_{3\,g}q_2{}^2-57.42\,q_2\,\alpha_3 N-3.98\,\alpha_{2\,g}q_2\,V_{12}{}^2-2119.14\\
&q_2{}^3\alpha_3\alpha_2+454.01\,\alpha_3 q_2{}^2\alpha_2+500.56-1.0\,V_{1g}{}^2\beta_{3\,g}-511.42\,q_2{}^3\alpha_3\alpha_1 q_1{}^2\\
&+1041.22\,\alpha_3 q_2{}^2\alpha_1 q_1{}^2-706.38\,q_2\,\alpha_3\alpha_1 q_1{}^2+1.0\,\alpha_{2\,g}V_{12}{}^2-2.65\,\alpha_{2\,g}q_2{}^3\\
&V_{10}{}^2+2.65\,V_{1g}{}^2\beta_{3\,g}q_2{}^3+1.0\,\alpha_{2\,g}V_{10}{}^2-2.0\,V_{1g}{}^2\beta_g+12.30\,\alpha_3 N\\
n_{1o} =& -706.38\,\alpha_3 q_2\,\alpha_2+1041.22\,\alpha_3 q_2{}^2\alpha_2+151.34\,\alpha_3\alpha_2-511.42\,q_2{}^3\alpha_3\alpha_2\\
n_{2o} =& -1534.26\,q_2{}^4\alpha_3\alpha_2-2119.14\,\alpha_3 q_2{}^2\alpha_2+3123.66\,q_2{}^3\alpha_3\alpha_2\\
&+454.01\,\alpha_3 q_2\,\alpha_2\\
n_{3o} =& \;151.34\,\alpha_3\alpha_1-511.42\,q_2{}^3\alpha_3\alpha_1-706.38\,q_2\,\alpha_3\alpha_1+1041.22\,\alpha_3 q_2{}^2\alpha_1\\
n_{4o} =& -1412.76\,\alpha_3 q_2\,\alpha_1 q_1-1022.84\,q_2{}^3\alpha_3\alpha_1 q_1+302.67\,\alpha_3\alpha_1 q_1\\
&+2082.44\,q_2{}^2\alpha_3\alpha_1 q_1\\
n_{5o} =& -2.0\,\beta_g-1.0\,\beta_{3\,g}+3.98\,\beta_{3\,g}q_2+2.65\,\beta_{3\,g}q_2{}^3-5.56\,\beta_{3\,g}q_2{}^2\\
n_{6o} =& \;5.56\,\alpha_{2\,g}q_2{}^2-2.55\,\alpha_{2\,g}q_2{}^3-3.98\,\alpha_{2\,g}q_2+1.0\,\alpha_{2\,g}
\end{aligned}
$$

$$
\begin{aligned}
n_{7o} =& \;151.34\,q_2\,\alpha_3\alpha_1-706.38\,\alpha_3 q_2{}^2\alpha_1+1041.22\,q_2{}^3\alpha_3\alpha_1-511.42\,q_2{}^4\alpha_3\alpha_1\\
n_{8o} =& -1412.76\,q_2{}^2\alpha_3\alpha_1 q_1+302.67\,\alpha_3 q_2\,\alpha_1 q_1+2082.44\,q_2{}^3\alpha_3\alpha_1 q_1\\
&-1022.84\,q_2{}^4\alpha_3\alpha_1 q_1\\
n_{9o} =& \;2.65\,\beta_{3\,g}q_2{}^4-7.41\,\beta_{3\,g}q_2{}^3+7.98\,\beta_{3\,g}q_2{}^2-4.0\,\beta_{3\,g}q_2\\
&+0.83\,\beta_{3\,g}-0.83\,\beta_g+1.0\,\beta_g q_2\\
n_{10o} =& -1.85\,\alpha_g q_2{}^3+1.0\,\alpha_{2\,g}q_2-2.55\,\alpha_{2\,g}q_2{}^4-3.0\,\alpha_g q_2+0.83\,\alpha_g+5.56\\
&\alpha_{2\,g}q_2{}^3-3.98\,\alpha_{2\,g}q_2{}^2+3.98\,\alpha_g q_2{}^2\\
n_{11o} =& -3.98\,q2\,c_3-2.65\,q2^3c_3+1.0\,c_2+5.56\,q2^2c_3
\end{aligned}
$$

$$
\lambda_2=\sqrt{\frac{k_o}{m_o}},\quad s_1=\frac{n_{1o}}{m_o},\quad s_2=\frac{n_{2o}}{m_o},\quad s_3=\frac{n_{3o}}{m_o},\quad s_4=\frac{n_{4o}}{m_o},\quad s_5=\frac{n_{5o}}{m_o},
$$
$$
s_6=\frac{n_{6o}}{m_o},\quad s_7=\frac{n_{7o}}{m_o},\quad s_8=\frac{n_{8o}}{m_o},\quad s_9=\frac{n_{9o}}{m_o},\quad s_{10}=\frac{n_{10o}}{m_o},\quad s_{11}=\frac{n_{11o}}{m_o}
$$

Appendix B

$B_1=(2\,k_2t_{11}-2\,k_1s_{11})\,\omega_1$, $B_2=(2\,k_2t_{11}-2\,k_2s_{11})\,\omega_2$, $B_3=s_9\eta_{21}+s_{10}(\eta_{11}+\eta_{12})-k_2\,(t_9\eta_{11}+t_{10}\eta_{12})$ $B_4=4\,\omega_1\,(k_1-k_2)$ $B_5=(2\,k_2s_{11}-2\,k_1t_{11})\,\omega_2$, $B_6=(2\,k_1s_{11}-2\,k_1t_{11})\,\omega_1$, $B_7=k_1\,(t_9\eta_{11}+t_{10}\eta_{12})-s_9\eta_{21}-s_{10}\,(\eta_{11}+\eta_{12})$, $B_8=4\,\omega_2\,(k_1-k_2)$.

$$c_{11} = -\frac{-{\lambda_2}^2 t_2 + t_8 s_2 {k_1}^2 + t_8 s_4 k_1 + t_8 s_7 + 4\,{\omega_1}^2 t_2 + 4\,{\omega_1}^2 t_4 k_1 + 4\,{\omega_1}^2 t_7 {k_1}^2 - {\lambda_2}^2 t_7 {k_1}^2 - {\lambda_2}^2 t_4 k_1}{t_8 s_8 - 16\,{\omega_1}^4 + 4\,{\omega_1}^2 {\lambda_1}^2 + 4\,{\lambda_2}^2 {\omega_1}^2 - {\lambda_2}^2 {\lambda_1}^2}$$
$$c_{12} = -2\,\frac{-{\lambda_2}^2 t_2 + t_8 s_2 {k_1}^2 + t_8 s_4 k_1 + t_8 s_7 - {\lambda_2}^2 t_4 k_1 - {\lambda_2}^2 t_7 {k_1}^2}{t_8 s_8 - {\lambda_2}^2 {\lambda_1}^2}$$
$$c_{13} = -\frac{-{\lambda_2}^2 t_2 + t_8 s_2 {k_2}^2 + t_8 s_4 k_2 + t_8 s_7 + 4\,{\omega_2}^2 t_2 + 4\,{\omega_2}^2 t_4 k_2 + 4\,{\omega_2}^2 t_7 {k_2}^2 - {\lambda_2}^2 t_7 {k_2}^2 - {\lambda_2}^2 t_4 k_2}{t_8 s_8 - 16\,{\omega_2}^4 + 4\,{\omega_2}^2 {\lambda_1}^2 + 4\,{\lambda_2}^2 {\omega_2}^2 - {\lambda_2}^2 {\lambda_1}^2}$$
$$c_{14} = -2\,\frac{-{\lambda_2}^2 t_2 + t_8 s_2 {k_2}^2 + t_8 s_4 k_2 + t_8 s_7 - {\lambda_2}^2 t_4 k_2 - {\lambda_2}^2 t_7 {k_2}^2}{t_8 s_8 - {\lambda_2}^2 {\lambda_1}^2}, \quad c_{15} = \frac{c_{15N}}{c_{15D}}, \quad c_{16} = \frac{c_{16N}}{c_{16D}}$$

$$\begin{aligned} c_{15N} =& -2\,t_8 s_2 k_1 k_2 + 2\,{\omega_1}^2 t_7 k_1 k_2 + {\omega_2}^2 t_4 k_2 - 2\,\omega_1 \omega_2 t_4 k_1 - 2\,\omega_1 \omega_2 t_4 k_2 + 2\,{\omega_2}^2 \\ & t_7 k_1 k_2 + 2\,{\omega_1}^2 t_2 - 2\,{\lambda_2}^2 t_7 k_1 k_2 - 4\,\omega_1 \omega_2 t_7 k_1 k_2 - {\lambda_2}^2 t_4 k_2 + t_8 s_4 k_1 \\ & - {\lambda_2}^2 t_4 k_1 + t_8 s_4 k_2 + 2\,t_8 s_7 + {\omega_1}^2 t_4 k_1 - 4\,\omega_1 \omega_2 t_2 + {\omega_2}^2 t_4 k_1 \\ & + {\omega_1}^2 t_4 k_2 - 2\,{\lambda_2}^2 t_2 + 2\,{\omega_2}^2 t_2 \\ c_{15D} =& -2\,\omega_1 \omega_2 {\lambda_1}^2 - 2\,{\lambda_2}^2 \omega_1 \omega_2 + 4\,{\omega_1}^3 \omega_2 + t_8 s_8 - {\omega_2}^4 - {\omega_1}^4 - 6\,{\omega_1}^2 {\omega_2}^2 \\ & + {\omega_1}^2 {\lambda_1}^2 + 4\,\omega_1 {\omega_2}^3 + {\omega_2}^2 {\lambda_1}^2 + {\lambda_2}^2 {\omega_1}^2 + {\lambda_2}^2 {\omega_2}^2 - {\lambda_2}^2 {\lambda_1}^2 \\ c_{16N} =& -2\,t_8 s_2 k_1 k_2 + 2\,{\omega_1}^2 t_7 k_1 k_2 + 2\,\omega_1 \omega_2 t_4 k_1 + 2\,\omega_1 \omega_2 t_4 k_2 + 2\,{\omega_2}^2 t_7 k_1 k_2 \\ & - 2\,{\lambda_2}^2 t_7 k_1 k_2 - {\lambda_2}^2 t_4 k_1 + t_8 s_4 k_2 + 4\,\omega_1 \omega_2 t_7 k_1 k_2 + t_8 s_4 k_1 \\ & + {\omega_2}^2 t_4 k_2 + 2\,t_8 s_7 - {\lambda_2}^2 t_4 k_2 + {\omega_1}^2 t_4 k_1 + 2\,{\omega_2}^2 t_2 + {\omega_2}^2 t_4 k_1 \\ & - 2\,{\lambda_2}^2 t_2 + 2\,{\omega_1}^2 t_2 + 4\,\omega_1 \omega_2 t_2 + {\omega_1}^2 t_4 k_2 \end{aligned}$$

$$\begin{aligned} c_{16D} =& \quad 2\,\omega_1 \omega_2 {\lambda_1}^2 + 2\,{\lambda_2}^2 \omega_1 \omega_2 - {\omega_1}^4 - {\omega_2}^4 + t_8 s_8 - 4\,{\omega_1}^3 \omega_2 - 6\,{\omega_1}^2 {\omega_2}^2 \\ & + {\omega_1}^2 {\lambda_1}^2 - 4\,\omega_1 {\omega_2}^3 + {\omega_2}^2 {\lambda_1}^2 + {\lambda_2}^2 {\omega_1}^2 + {\lambda_2}^2 {\omega_2}^2 - {\lambda_2}^2 {\lambda_1}^2 \end{aligned}$$

$$c_{21} = \frac{{\lambda_1}^2 s_7 - t_2 s_8 - 4\,{\omega_1}^2 s_7 - 4\,{\omega_1}^2 s_4 k_1 + {\lambda_1}^2 s_4 k_1 - t_4 k_1 s_8 - 4\,{\omega_1}^2 s_2 {k_1}^2 + {\lambda_1}^2 s_2 {k_1}^2 - t_7 {k_1}^2 s_8}{t_8 s_8 - 16\,{\omega_1}^4 + 4\,{\omega_1}^2 {\lambda_1}^2 + 4\,{\lambda_2}^2 {\omega_1}^2 - {\lambda_2}^2 {\lambda_1}^2}$$
$$c_{22} = 2\,\frac{{\lambda_1}^2 s_2 {k_1}^2 + {\lambda_1}^2 s_4 k_1 + {\lambda_1}^2 s_7 - t_2 s_8 - t_4 k_1 s_8 - t_7 {k_1}^2 s_8}{t_8 s_8 - {\lambda_2}^2 {\lambda_1}^2}$$
$$c_{23} = \frac{-4\,{\omega_2}^2 s_7 + {\lambda_1}^2 s_7 - t_2 s_8 - 4\,{\omega_2}^2 s_4 k_2 + {\lambda_1}^2 s_4 k_2 - t_4 k_2 s_8 - 4\,{\omega_2}^2 s_2 {k_2}^2 + {\lambda_1}^2 s_2 {k_2}^2 - t_7 {k_2}^2 s_8}{t_8 s_8 - 16\,{\omega_2}^4 + 4\,{\omega_2}^2 {\lambda_1}^2 + 4\,{\lambda_2}^2 {\omega_2}^2 - {\lambda_2}^2 {\lambda_1}^2}$$
$$c_{24} = 2\,\frac{{\lambda_1}^2 s_2 {k_2}^2 + {\lambda_1}^2 s_4 k_2 + {\lambda_1}^2 s_7 - t_2 s_8 - t_4 k_2 s_8 - t_7 {k_2}^2 s_8}{t_8 s_8 - {\lambda_2}^2 {\lambda_1}^2}, \quad c_{25} = \frac{c_{25N}}{c_{25D}}, \quad c_{26} = \frac{c_{26N}}{c_{26D}}$$

$$\begin{aligned}
c_{25N} =& -2\,\omega_1{}^2 s_2 k_1 k_2 + 2\,\omega_1\omega_2 s_4 k_2 + 2\,\omega_1\omega_2 s_4 k_1 - 2\,\omega_2{}^2 s_2 k_1 k_2 + 2\,\lambda_1{}^2 s_2 k_1 k_2 - \\
& 2\,t_7 k_1 k_2 s_8 - 2\,\omega_2{}^2 s_7 + 2\,\lambda_1{}^2 s_7 - 2\,t_2 s_8 - 2\,\omega_1{}^2 s_7 - \omega_1{}^2 s_4 k_2 - \omega_1{}^2 s_4 k_1 \\
& + 4\,\omega_1\omega_2 s_7 - \omega_2{}^2 s_4 k_2 - \omega_2{}^2 s_4 k_1 + \lambda_1{}^2 s_4 k_2 + \lambda_1{}^2 s_4 k_1 - t_4 k_1 s_8 - t_4 k_2 s_8 \\
& + 4\,\omega_1\omega_2 s_2 k_1 k_2 \\
c_{25D} =& -2\,\omega_1\omega_2\lambda_1{}^2 - 2\,\lambda_2{}^2\omega_1\omega_2 + 4\,\omega_1{}^3\omega_2 + t_8 s_8 - \omega_2{}^4 - \omega_1{}^4 - 6\,\omega_1{}^2\omega_2{}^2 \\
& + \omega_1{}^2\lambda_1{}^2 + 4\,\omega_1\omega_2{}^3 + \omega_2{}^2\lambda_1{}^2 + \lambda_2{}^2\omega_1{}^2 + \lambda_2{}^2\omega_2{}^2 - \lambda_2{}^2\lambda_1{}^2 \\
c_{26N} =& -2\,\omega_1{}^2 s_2 k_1 k_2 - 2\,\omega_1\omega_2 s_4 k_2 - 2\,\omega_1\omega_2 s_4 k_1 - 2\,\omega_2{}^2 s_2 k_1 k_2 + 2\,\lambda_1{}^2 s_2 k_1 k_2 \\
& - 2\,t_7 k_1 k_2 s_8 - 2\,\omega_2{}^2 s_7 + 2\,\lambda_1{}^2 s_7 - 2\,t_2 s_8 - 2\,\omega_1{}^2 s_7 - \omega_1{}^2 s_4 k_2 - \omega_1{}^2 s_4 k_1 \\
& - 4\,\omega_1\omega_2 s_7 - \omega_2{}^2 s_4 k_2 - \omega_2{}^2 s_4 k_1 + \lambda_1{}^2 s_4 k_2 + \lambda_1{}^2 s_4 k_1 - t_4 k_1 s_8 - t_4 k_2 s_8 \\
& - 4\,\omega_1\omega_2 s_2 k_1 k_2 \\
c_{26D} =& \;\; 2\,\omega_1\omega_2\lambda_1{}^2 + 2\,\lambda_2{}^2\omega_1\omega_2 - \omega_1{}^4 - \omega_2{}^4 + t_8 s_8 - 4\,\omega_1{}^3\omega_2 - 6\,\omega_1{}^2\omega_2{}^2 \\
& + \omega_1{}^2\lambda_1{}^2 - 4\,\omega_1\omega_2{}^3 + \omega_2{}^2\lambda_1{}^2 + \lambda_2{}^2\omega_1{}^2 + \lambda_2{}^2\omega_2{}^2 - \lambda_2{}^2\lambda_1{}^2
\end{aligned}$$

$$\begin{aligned}
g_{11} =& \, 2\,t_2 c_{15} + 2\,t_2 c_{11} + t_3 k_1{}^2 + t_4 c_{21} + t_4 c_{25} + t_4 k_1 c_{15} + 2\,t_3 k_1 k_2 + t_4 k_2 c_{11} \\
& + 2\,t_7 k_1 c_{25} + 2\,t_7 k_2 c_{21} + 3\,t_1 \\
g_{12} =& \, t_4 k_2 c_{13} + 3\,t_1 + 2\,t_2 c_{13} + 3\,t_3 k_2{}^2 + 2\,t_2 c_{14} + t_4 k_2 c_{14} + 2\,t_7 k_2 c_{24} + 2\,t_7 k_2 c_{23} \\
& + t_4 c_{24} + t_4 c_{23} \\
g_{13} =& \, 2\,t_7 k_2 c_{22} + t_4 c_{22} + t_4 c_{26} + t_4 k_2 c_{12} + 2\,t_2 c_{16} + t_4 k_1 c_{16} + 2\,t_2 c_{12} + 4\,t_3 k_1 k_2 \\
& + 2\,t_3 k_1{}^2 + 6\,t_1 + 2\,t_7 k_1 c_{26} \\
g_{14} =& \, 2\,t_2 c_{13} + t_3 k_2{}^2 + t_4 c_{23} + 2\,t_3 k_1 k_2 + t_4 k_1 c_{13} + 2\,t_7 k_1 c_{23} + 3\,t_1 \\
g_{15} =& \, 2\,t_7 k_1 c_{22} + 2\,t_7 k_1 c_{21} + t_4 k_1 c_{12} + 2\,t_2 c_{12} + t_4 c_{22} + t_4 k_1 c_{11} + 3\,t_3 k_1{}^2 \\
& + t_4 c_{21} + 3\,t_1 + 2\,t_2 c_{11} \\
g_{16} =& \, t_4 k_1 c_{14} + 2\,t_7 k_1 c_{24} + t_4 k_2 c_{15} + t_4 k_2 c_{16} + 2\,t_7 k_2 c_{25} + 4\,t_3 k_1 k_2 + 2\,t_7 k_2 c_{26} \\
& + 2\,t_3 k_2{}^2 + t_4 c_{25} + t_4 c_{26} + 2\,t_2 c_{14} + 2\,t_2 c_{16} + t_4 c_{24} + 2\,t_2 c_{15} + 6\,t_1 \\
f_{11} =& \, 2\,t_2 c_{13} + t_3 k_2{}^2 + t_4 c_{23} + 2\,t_3 k_1 k_2 + t_4 k_1 c_{13} + 2\,t_7 k_1 c_{23} + 3\,t_1 \\
f_{12} =& \, 2\,t_7 k_1 c_{22} + 2\,t_7 k_1 c_{21} + t_4 k_1 c_{12} + 2\,t_2 c_{12} + t_4 c_{22} + t_4 k_1 c_{11} + 3\,t_3 k_1{}^2 + t_4 c_{21} \\
& + 3\,t_1 + 2\,t_2 c_{11} \\
f_{13} =& \, t_4 k_1 c_{14} + 2\,t_7 k_1 c_{24} + t_4 k_2 c_{15} + t_4 k_2 c_{16} + 2\,t_7 k_2 c_{25} + 4\,t_3 k_1 k_2 + 2\,t_7 k_2 c_{26} \\
& + 2\,t_3 k_2{}^2 + t_4 c_{25} + t_4 c_{26} + 2\,t_2 c_{14} + 2\,t_2 c_{16} + t_4 c_{24} + 2\,t_2 c_{15} + 6\,t_1
\end{aligned}$$

$$\begin{aligned}
f_{14} =& \, 2\,t_2 c_{15} + 2\,t_2 c_{11} + t_3 k_1{}^2 + t_4 c_{21} + t_4 c_{25} + t_4 k_1 c_{15} + 2\,t_3 k_1 k_2 + t_4 k_2 c_{11} \\
& + 2\,t_7 k_1 c_{25} + 2\,t_7 k_2 c_{21} + 3\,t_1 \\
f_{15} =& \, t_4 k_2 c_{13} + 3\,t_1 + 2\,t_2 c_{13} + 3\,t_3 k_2{}^2 + 2\,t_2 c_{14} + t_4 k_2 c_{14} + 2\,t_7 k_2 c_{24} + 2\,t_7 k_2 c_{23} \\
& + t_4 c_{24} + t_4 c_{23} \\
f_{16} =& \, 2\,t_7 k_2 c_{22} + t_4 c_{22} + t_4 c_{26} + t_4 k_2 c_{12} + 2\,t_2 c_{16} + t_4 k_1 c_{16} + 2\,t_2 c_{12} + 4\,t_3 k_1 k_2 \\
& + 2\,t_3 k_1{}^2 + 6\,t_1 + 2\,t_7 k_1 c_{26}
\end{aligned}$$

$$\begin{aligned}
g_{21} &= s_4c_{21} + s_3k_2 + 2\,s_3k_1 + s_4c_{25} + 2\,s_7c_{15} + 3\,s_1{k_1}^2k_2 + s_4c_{15}k_1 + 2\,s_2k_1c_{25} \\
&\quad + 2\,s_2k_2c_{21} + 2\,s_7c_{11} + s_4c_{11}k_2 \\
g_{22} &= 2\,s_2k_2c_{24} + 3\,s_3k_2 + 2\,s_7c_{14} + s_4c_{24} + s_4c_{23} + s_4c_{13}k_2 + 2\,s_2k_2c_{23} + s_4 \\
&\quad c_{14}k_2 + 2\,s_7c_{13} + 3\,s_1{k_2}^3 \\
g_{23} &= s_4c_{12}k_2 + 2\,s_3k_2 + 2\,s_7c_{12} + 6\,s_1{k_1}^2k_2 + 2\,s_7c_{16} + s_4c_{22} + 4\,s_3k_1 + 2\,s_2 \\
&\quad k_1c_{26} + s_4c_{26} + 2\,s_2k_2c_{22} + s_4c_{16}k_1 \\
g_{24} &= 2\,s_2k_1c_{23} + 3\,s_1{k_2}^2k_1 + s_4c_{13}k_1 + s_4c_{23} + 2\,s_7c_{13} + s_3k_1 + 2\,s_3k_2 \\
g_{25} &= 2\,s_7c_{12} + 2\,s_2k_1c_{22} + 3\,s_3k_1 + s_4c_{12}k_1 + s_4c_{11}k_1 + 2\,s_2k_1c_{21} + s_4c_{22} \\
&\quad + s_4c_{21} + 2\,s_7c_{11} + 3\,s_1{k_1}^3 \\
g_{26} &= s_4c_{14}k_1 + 6\,s_1{k_2}^2k_1 + 2\,s_2k_1c_{24} + s_4c_{15}k_2 + s_4c_{16}k_2 + 2\,s_2k_2c_{25} \\
&\quad + 2\,s_2k_2c_{26} + 2\,s_7c_{14} + 2\,s_7c_{15} + 4\,s_3k_2 + 2\,s_3k_1 + 2\,s_7c_{16} + s_4c_{24} \\
&\quad + s_4c_{26} + s_4c_{25} \\
f_{21} &= 2\,s_2k_1c_{23} + 3\,s_1{k_2}^2k_1 + s_4c_{13}k_1 + s_4c_{23} + 2\,s_7c_{13} + s_3k_1 + 2\,s_3k_2 \\
f_{22} &= 2\,s_7c_{12} + 2\,s_2k_1c_{22} + 3\,s_3k_1 + s_4c_{12}k_1 + s_4c_{11}k_1 + 2\,s_2k_1c_{21} + s_4c_{22} \\
&\quad + s_4c_{21} + 2\,s_7c_{11} + 3\,s_1{k_1}^3 \\
f_{23} &= s_4c_{14}k_1 + 6\,s_1{k_2}^2k_1 + 2\,s_2k_1c_{24} + s_4c_{15}k_2 + s_4c_{16}k_2 + 2\,s_2k_2c_{25} \\
&\quad + 2\,s_2k_2c_{26} + 2\,s_7c_{14} + 2\,s_7c_{15} + 4\,s_3k_2 + 2\,s_3k_1 + 2\,s_7c_{16} + s_4c_{24} \\
&\quad + s_4c_{26} + s_4c_{25} \\
f_{24} &= s_4c_{21} + s_3k_2 + 2\,s_3k_1 + s_4c_{25} + 2\,s_7c_{15} + 3\,s_1{k_1}^2k_2 + s_4c_{15}k_1 \\
&\quad + 2\,s_2k_1c_{25} + 2\,s_2k_2c_{21} + 2\,s_7c_{11} + s_4c_{11}k_2 \\
f_{25} &= 2\,s_2k_2c_{24} + 3\,s_3k_2 + 2\,s_7c_{14} + s_4c_{24} + s_4c_{23} + s_4c_{13}k_2 + 2\,s_2k_2c_{23} \\
&\quad + s_4c_{14}k_2 + 2\,s_7c_{13} + 3\,s_1{k_2}^3 \\
f_{26} &= s_4c_{12}k_2 + 2\,s_3k_2 + 2\,s_7c_{12} + 6\,s_1{k_1}^2k_2 + 2\,s_7c_{16} + s_4c_{22} + 4\,s_3k_1 \\
&\quad + 2\,s_2k_1c_{26} + s_4c_{26} \\
&\quad + 2\,s_2k_2c_{22} + s_4c_{16}k_1
\end{aligned}$$

$$\begin{gathered}
B_{11} = 1 + k_1\bar{k_1}, \quad B_{12} = 1 + k_2\bar{k_1}, \quad B_{13} = t_{11} + s_{11}k_1\bar{k_1}, \\
B_{14} = t_{11} + s_{11}k_2\bar{k_1}, \quad G_{11} = 1 + k_1\bar{k_2}, \quad G_{12} = 1 + k_2\bar{k_2}, \\
G_{13} = t_{11} + s_{11}k_2\bar{k_2}, \quad G_{14} = t_{11} + s_{11}k_1\bar{k_2}
\end{gathered}$$

$$\begin{gathered}
g_1 = g_{11} + \bar{k_1}g_{21}, \quad g_2 = g_{12} + \bar{k_1}g_{22}, \quad g_3 = g_{13} + \bar{k_1}g_{23}, \\
g_4 = g_{14} + \bar{k_1}g_{24}, \quad g_5 = g_{15} + \bar{k_1}g_{25}, \quad g_6 = g_{16} + \bar{k_1}g_{26}, \\
f_1 = f_{11} + \bar{k_2}f_{21}, \quad f_2 = f_{12} + \bar{k_2}f_{22}, \quad f_3 = f_{13} + \bar{k_2}f_{23},
\end{gathered}$$

$$f_4 = f_{14} + \bar{k_2}f_{24}, \quad f_5 = f_{15} + \bar{k_2}f_{25}, \quad f_6 = f_{16} + \bar{k_2}f_{26}$$

$$\begin{aligned}
\bar{G} &= \bar{g_1}\bar{A}_2A_1^2e^{-i\sigma_2T_1} + \bar{g_2}\bar{A}_2A_2^2e^{i\sigma_2T_1} + \bar{g_3}A_1A_2\bar{A}_1e^{i\sigma_2T_1} + \bar{g_4}A_2^2\bar{A}_1e^{2i\sigma_2T_1} \\
&\quad + \bar{g_5}A_1^2\bar{A}_1 + \bar{g_6}A_1\bar{A}_2A_2; \\
\bar{F} &= \bar{f_1}\bar{A}_1A_2^2e^{i\sigma_2T_1} + \bar{f_2}\bar{A}_1A_1^2e^{-i\sigma_2T_1} + \bar{f_3}A_1A_2\bar{A}_2e^{-i\sigma_2T_1} + \bar{f_4}A_1^2\bar{A}_2e^{-2i\sigma_2T_1} \\
&\quad + \bar{f_5}A_2^2\bar{A}_2 + \bar{f_6}A_1\bar{A}_1A_2;
\end{aligned}$$

$$h_{11} = 1/4\,\frac{B_{13}G_{12} - B_{12}G_{14}}{\omega_1\,(B_{11}G_{12} - B_{12}G_{11})}, \quad h_{22} = 1/2\,\frac{B_1}{B_4}, \quad h_{33} = 1/2\,\frac{B_2}{B_4}, \quad h_{44} = \frac{B_3}{B_4},$$
$$h_{55} = 1/4\,\frac{B_{14}G_{12} - B_{12}G_{13}}{\omega_1\,(B_{11}G_{12} - B_{12}G_{11})}, \quad h_{66} = 1/2\,\frac{B_5}{B_8}, \quad h_{77} = 1/2\,\frac{B_6}{B_8}, \quad h_{88} = \frac{B_7}{B_8},$$
$$h_{99} = 1/16\,\frac{G_{12}}{\omega_1\,(B_{11}G_{12} - B_{12}G_{11})}, \quad h_{1010} = 1/16\,\frac{B_{12}}{\omega_1\,(B_{11}G_{12} - B_{12}G_{11})}$$
$$h_1 = h_{11},\, h_2 = 2h_{22},\, h_3 = 2h_{33},\, h_4 = 2h_{44},\, h_5 = 2h_{55},\, h_6 = h_{66},\, h_7 = h_{77},$$
$$h_8 = h_{88},\, h_9 = 2h_{99}, h_{10} = 2h_{1010}$$

$$l_{11} = 1/4\,\frac{B_{14}G_{11} - B_{11}G_{13}}{\omega_2\,(B_{12}G_{11} - B_{11}G_{12})}, l_{22} = 1/2\,\frac{B_5}{B_8}, \quad l_{33} = 1/2\,\frac{B_6}{B_8}, \quad l_{44} = \frac{B_7}{B_8}$$
$$l_{55} = 1/4\,\frac{B_{13}G_{11} - B_{11}G_{14}}{\omega_2\,(B_{12}G_{11} - B_{11}G_{12})}, \quad l_{66} = 1/2\,\frac{B_1}{B_4}, \quad l_{77} = 1/2\,\frac{B_2}{B_4}, \quad l_{88} = \frac{B_3}{B_4},$$
$$l_{99} = 1/16\,\frac{G_{11}}{\omega_2\,(B_{12}G_{11} - B_{11}G_{12})}, \quad l_{1010} = 1/16\,\frac{B_{11}}{\omega_2\,(B_{12}G_{11} - B_{11}G_{12})}$$
$$l_1 = l_{11},\, l_2 = 2l_{22},\, l_3 = 2l_{33},\, l_4 = 2l_{44},\, l_5 = 2l_{55},\, l_6 = l_{66},\, l_7 = l_{77}, l_8 = l_{88},$$
$$l_9 = 2l_{99}, l_{10} = 2l_{1010}$$

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