Helmholtz' Theorem

EE 141 Lecture Notes
Topic 3

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Motivation

HE THOUGHT HE WAS LICKED-THEN
A TIP GOT BILL A GOOD JOB!
MY RAISE DIDN'T COME THROUGH MARY—I MIGHT AS WELL GIVE UP. IT ALL LOOKS SO HOPELESS.
IT ISN'T HOPELESS EITHER BILL. WHY DON'T YOU TRY A NEW FIELD LIKE RADIO?
TOM GREEN WENT INTO RADIO AND HE'S MAKING GOOD MONEY, TOO. I'LL SEE HIM RIGHT AWAY.
BILL, JUST MAILING THAT COUPON GAVE ME A QUICK START TO SUCCESS IN RADIO. MAIL THIS ONE TONIGHT
TOM'S RIGHT—AN UNTRAINED MAN HASN'T A CHANCE. I'M GOING TO TRAIN FOR RADIO TOO. IT'S TODAY'S FIELD OF GOOD PAY OPPORTUNITIES
U.S. MAIL
TRAINING FOR RADIO IS EASY AND I'M GETTING ALONG FAST...
SOON I CAN GET A JOB SERVICING SETS.
OR INSTALLING LOUD SPEAKER SYSTEMS
OR IN A BROADCASTING STATION
THERE'S NO END TO THE GOOD JOBS FOR THE TRAINED RADIO MAN
YOU SURE KNOW RADIO—MY SET NEVER SOUNDED BETTER
THAT'S $15 I'VE MADE THIS WEEK IN SPARE TIME
THANKS!
I HAVE A GOOD FULL TIME RADIO JOB NOW--AND A BRIGHT FUTURE AHEAD IN RADIO
OH BILL, IT'S WONDERFUL YOU'VE GONE AHEAD SO FAST IN RADIO.

Helmholtz' Theorem

Because

$$\nabla^2 \left(\frac{1}{R} \right) = -4\pi\delta(\mathbf{R}) \tag{1}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ with magnitude $R = |\mathbf{R}|$ and where

$$\delta(\mathbf{R}) = \delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$$

is the three-dimensional Dirac delta function, then any sufficiently well-behaved vector function $\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z)$ can be represented as

$$\begin{aligned} \mathbf{F}(\mathbf{r}) &= \int_V \mathbf{F}(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')\, d^3r' = -\frac{1}{4\pi} \int_V \mathbf{F}(\mathbf{r}')\nabla^2 \left(\frac{1}{R} \right) d^3r' \\ &= -\frac{1}{4\pi} \nabla^2 \int_V \frac{\mathbf{F}(\mathbf{r}')}{R}\, d^3r', \end{aligned} \tag{2}$$

the integration extending over any region V that contains the point $\mathbf{r}$.

Helmholtz' Theorem

With the identity $\nabla \times \nabla \times = \nabla\nabla \cdot - \nabla^2$, Eq. (2) may be written as

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi}\nabla \times \nabla \times \int_V \frac{\mathbf{F}(\mathbf{r}')}{R} d^3r' - \frac{1}{4\pi}\nabla\nabla \cdot \int_V \frac{\mathbf{F}(\mathbf{r}')}{R} d^3r'. \tag{3}$$

Consider first the divergence term appearing in this expression. Because the vector differential operator ∇ does not operate on the primed coordinates, then

$$\frac{1}{4\pi}\nabla \cdot \int_V \frac{\mathbf{F}(\mathbf{r}')}{R} d^3r' = \frac{1}{4\pi}\int_V \mathbf{F}(\mathbf{r}') \cdot \nabla \left(\frac{1}{R}\right) d^3r'. \tag{4}$$

Helmholtz' Theorem

The integrand appearing in this expression may be expressed as

$$\begin{aligned}\mathbf{F}(\mathbf{r}')\cdot\nabla\left(\frac{1}{R}\right) &= -\mathbf{F}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{R}\right)\\ &= -\nabla'\cdot\left(\frac{\mathbf{F}(\mathbf{r}')}{R}\right)+\frac{1}{R}\nabla'\cdot\mathbf{F}(\mathbf{r}'),\end{aligned}\qquad(5)$$

where the prime on ∇' denotes differentiation with respect to the primed coordinates alone, viz.

$$\nabla' = \hat{\mathbf{1}}_x\frac{\partial}{\partial x'}+\hat{\mathbf{1}}_y\frac{\partial}{\partial y'}+\hat{\mathbf{1}}_z\frac{\partial}{\partial z'}$$

when $\hat{\mathbf{1}}_{j'}=\hat{\mathbf{1}}_j$, $j=x,y,z$.

Helmholtz' Theorem

Substitution of Eq. (5) into Eq. (4) and application of the divergence theorem to the first term then yields

$$\begin{aligned}
\frac{1}{4\pi}\nabla\cdot\int_V \frac{\mathbf{F}(\mathbf{r}')}{R}d^3r' &= -\frac{1}{4\pi}\int_V \nabla'\cdot\left(\frac{\mathbf{F}(\mathbf{r}')}{R}\right)d^3r' \\
&\qquad +\frac{1}{4\pi}\int_V \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{R}d^3r' \\
&= -\frac{1}{4\pi}\oint_S \frac{1}{R}\mathbf{F}(\mathbf{r}')\cdot\hat{\mathbf{n}}d^2r' \\
&\qquad +\frac{1}{4\pi}\int_V \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{R}d^3r' \\
&= \phi(\mathbf{r}), \qquad (6)
\end{aligned}$$

which is the desired form of the scalar potential $\phi(\mathbf{r})$ for the vector field $\mathbf{F}(\mathbf{r})$. Here S is the surface that encloses the regular region V containing the point $\mathbf{r}$.

Helmholtz' Theorem

For the curl term appearing in Eq. (3) one has that

$$\begin{aligned} \frac{1}{4\pi}\nabla \times \int_V \frac{\mathbf{F}(\mathbf{r}')}{R} d^3r' &= -\frac{1}{4\pi}\int_V \mathbf{F}(\mathbf{r}') \times \nabla\left(\frac{1}{R}\right) d^3r' \\ &= \frac{1}{4\pi}\int_V \mathbf{F}(\mathbf{r}') \times \nabla'\left(\frac{1}{R}\right) d^3r'. \end{aligned} \tag{7}$$

Moreover, the integrand appearing in the final form of the integral in Eq. (7) may be expressed as

$$\mathbf{F}(\mathbf{r}') \times \nabla'\left(\frac{1}{R}\right) = \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{R} - \nabla' \times \left(\frac{\mathbf{F}(\mathbf{r}')}{R}\right), \tag{8}$$

so that

Helmholtz' Theorem

$$
\begin{aligned}
\frac{1}{4\pi}\nabla \times \int_V \frac{\mathbf{F}(\mathbf{r}')}{R} d^3r' &= \frac{1}{4\pi}\int_V \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{R} d^3r' \\
&\quad - \frac{1}{4\pi}\int_V \nabla' \times \left(\frac{\mathbf{F}(\mathbf{r}')}{R}\right) d^3r' \\
&= \frac{1}{4\pi}\int_V \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{R} d^3r' \\
&\quad + \frac{1}{4\pi}\oint_S \frac{1}{R}\mathbf{F}(\mathbf{r}') \times \hat{\mathbf{n}} d^2r' \\
&= \mathbf{a}(\mathbf{r}), \qquad (9)
\end{aligned}
$$

which is the desired form of the vector potential.

Helmholtz' Theorem

Equations (3), (6), and (9) then show that

$$\boxed{\mathbf{F}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) + \nabla\times\mathbf{a}(\mathbf{r})} \tag{10}$$

where the scalar potential $\phi(\mathbf{r})$ is given by Eq. (6) and the vector potential $\mathbf{a}(\mathbf{r})$ by Eq. (9).

This expression may also be written as

$$\boxed{\mathbf{F}(\mathbf{r}) = \mathbf{F}_\ell(\mathbf{r}) + \mathbf{F}_t(\mathbf{r})} \tag{11}$$

known as the Helmholtz decomposition.

Helmholtz' Theorem

In the Helmholtz decomposition,

$$\begin{aligned}\mathbf{F}_\ell(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\frac{1}{4\pi}\nabla\int_V \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \frac{1}{4\pi}\nabla\oint_S \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\cdot\hat{\mathbf{n}}d^2r'\end{aligned} \tag{12}$$

is the longitudinal or irrotational part of the vector field (with $\nabla\times\mathbf{F}_\ell(\mathbf{r}') = \mathbf{0}$), and

$$\begin{aligned}\mathbf{F}_t(\mathbf{r}) &= \nabla\times\mathbf{a}(\mathbf{r}) = \frac{1}{4\pi}\nabla\times\nabla\times\int_V \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' \\ &= \frac{1}{4\pi}\nabla\times\int_V \frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \frac{1}{4\pi}\nabla\times\oint_S \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\times\hat{\mathbf{n}}d^2r'\end{aligned} \tag{13}$$

is the transverse or solenoidal part of the vector field (with $\nabla\cdot\mathbf{F}_\ell(\mathbf{r}') = 0$).

Helmholtz' Theorem

If the surface S recedes to infinity and if the vector field $\mathbf{F}(\mathbf{r})$ is regular at infinity, then the surface integrals appearing in Eqs. (12)–(13) become

$$\begin{aligned} \mathbf{F}_\ell(\mathbf{r}) &= -\nabla\phi(\mathbf{r}) \\ &= -\frac{1}{4\pi}\nabla\int_V \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r', \qquad (14) \\ \mathbf{F}_t(\mathbf{r}) &= \nabla\times\mathbf{a}(\mathbf{r}) \\ &= \frac{1}{4\pi}\nabla\times\int_V \frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'. \qquad (15) \end{aligned}$$

Taken together, the above results constitute what is known as Helmholtz' theorem or the Fundamental Theorem of Vector Calculus.

Helmholtz' Theorem

Theorem

Helmholtz' Theorem. *Let* $\mathbf{F}(\mathbf{r})$ *be any continuous vector field with continuous first partial derivatives. Then* $\mathbf{F}(\mathbf{r})$ *can be uniquely expressed in terms of the negative gradient of a scalar potential* $\phi(\mathbf{r})$ *& the curl of a vector potential* $\mathbf{a}(\mathbf{r})$, *as embodied in Eqs. (10)–(11).*

Hermann Ludwig Ferdinand von Helmholtz (1821–1894)