The Helmholtz Theorem: Applications and Discrepancies

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(Dated: December 2, 2008)

The Helmholtz theorem states that any vector with a divergence corresponding to a source density and a curl that corresponds to a circulation density, both vanishing at infinity, may be written as the sum of two parts: rotational and irrotational. Also known as the fundamental theorem of vector calculus, the Helmholtz Theorem has several useful applications in mathematics, mechanics, and electromagnetic theory; while its relevance is far-reaching, and the implications inherent to its use are even more wide spread, it is rarely understood fully and fails entirely when applied to expanded domains with complex boundaries. In fact, it is hard to justify Helmholtz's Theorem in all domains because it is so all-inclusive; while many scholars have tried to broaden the theorem's definition to include all implications, it is still a work in progress and needs to be expanded and appendicised further in order to encompass and fulfill all of its inherent properties beyond simple physical phenomenon.

I. INTRODUCTION

Helmholtz's Theorem is, essentially, nothing more or less than a decomposition theorem, a tool by which to break down a vector into separate parts that are more easily understood individually. Most of its forms are commented, however, because of their inability to work in expanded domains[1]. It was introduced by a German scientist named Hermann von Helmholtz in the mid-19th century, and scientists and mathematicians have been building on its foundation ever since. Helmholtz's theorem says that any vector field $\vec{F}$ that satisfies the conditions[3]

$$[\nabla \cdot \vec{F}]_{\infty} = 0 \tag{1}$$

$$[\nabla \times \vec{F}]_{\infty} = 0 \tag{2}$$

May be written as the sum of an irrotational and a solenoidal part:

$$\vec{F} = -\nabla\phi + \nabla \times \vec{A} \tag{3}$$

Here, $-\nabla\phi$ is the irrotational component and $\nabla \times \vec{A}$ is the rotational (solenoidal) part. The scalar potential is satisfied by

$$\phi(\vec{r}_1) = -\int_V \frac{\nabla \cdot \vec{F}}{4\pi|(\vec{r}' - \vec{r})|} d^3\vec{r}' \tag{4}$$

And the vector potential is

$$A(\vec{r}_1) = -\int_V \frac{\nabla \times \vec{F}}{4\pi|(\vec{r}' - \vec{r})|} d^3\vec{r}' \tag{5}$$

In any subjects involving simple and generalized vector fields, these equations hold true but, while Helmholtz's Theorem has become synonymous with the uniqueness theorem of a vector field, it does not explicitly indicate any uniqueness for bounded problems[1]; in fact, it only holds in simply connected domains without multiple surface boundaries, and does not allow for piecewise functions at all, which is why most forms of the theorem are incomplete[2]. According to the theorem, the Helmholtz equation is valid in all space, but this is physically and mathematically untrue.

II. PROPOSITIONS OF HELMHOLTZ'S THEOREM

In order for Helmholtz's Theorem to be satisfied, a vector must contain two parts: a solenoidal (or rotational) part and an irrotational part[3]. Simply stated, this means that a vector can be decomposed into the sum of the gradient of a scalar function and the curl of a vector function, as long as certain conditions hold true. It also must exist in a simply bound region of space. The propositions expanding Helmholtz's Theorem, which allow for it to be satisfied and the two parts (rotational and irrotational) to emerge, apply to the following vectors:[2]

(1) If $\vec{F}$ is a first-order vector function that is continuously differentiable with $|(\vec{r}' - \vec{r})|^2|\vec{F}|$ bounded at infinity, then it can be decomposed into solenoidal and irrotational parts. On the surface, this is equivalent to

$$\vec{n} \times \vec{F} = 0 \tag{6}$$

and

$$\vec{n} \cdot \vec{F} = 0 \tag{7}$$

with $\vec{n}$ representing a vector normal to the surface S.

(2) If $\vec{F}$ is a continuous second-order vector function that exists in all space with a surface integration or partial derivative tending to zero at infinity, then it satisfies the Helmholtz equation. A mathematical representation of this can be seen from equations (3), (4), and (5).

(3) A vector field function that is bound on the boundary in a limited domain can be decomposed.

(4) A vector $\vec{F}$ that satisfies a vector homogeneous boundary condition. This is similar to (2), and the mathematical representation is also found in equation (3). In this situation, the irrotational and solenoidal fields are mutually orthogonal and satisfy the same kind of boundary condition.

(5) An arbitrary vector function in a bounded domain V with a clearly defined divergence and curl of this domain and a given value on the boundary S has both a solenoidal and an irrotational part. This means that a solution to the following system is unique, and the uniqueness theorem is satisfied:

$$\nabla \times \vec{F(\vec{r})} = s(\vec{r}) \tag{8}$$

$$\nabla \cdot \vec{F(\vec{r})} = c(\vec{r}) \tag{9}$$

$$\vec{F(\vec{r})}|_r = \vec{F_0(\vec{r})}|_r \tag{10}$$

These are the various general conditions by which the Helmholtz theorem holds. They do not allow for any discontinuities or piece-wise functions, which makes them useful for theory but not as helpful for applied physics. According to Zhou, if a vector function is both irrotational and solenoidal then it must be a null vector function[1], but this statement was made without considering (1) above. The reason for this is that the statement was made before (1) was added, but because (1) has no known use outside of theory this statement is true for applied physical situations.

III. ZHOU'S DIRECT PROOF OF HELMHOLTZ'S GENERALIZED THEORUM[1]

The proof of Helmholtz's Theorem takes many forms dependent on the area it is trying to explain, but the proof of the generalized theorem is rooted in the following vector identity, which holds true for any second order differential:

$$\nabla^2 \vec{L} = \nabla(\nabla \cdot \vec{L}) - \nabla \times (\nabla \times \vec{L}) \tag{11}$$

In order to link this to a piecewise continuous function $\vec{F}(\vec{r})$, it must be stated that

$$\vec{F} = -\nabla(\nabla \cdot \vec{L}) - \nabla \times (\nabla \times \vec{L}) \tag{12}$$

Which leads to

$$\nabla^2 \vec{L}(\vec{r}) = -\vec{F}(\vec{r}) \tag{13}$$

This allows for various boundary conditions while still holding (2) and (3) true in all cases. $\vec{F}(\vec{r})$ can be broken down into its solenoidal and irrotational components by applying a corresponding Greene's function, and it is found that the irrotational component of $\vec{F}$ is

$$\vec{F_i}(\vec{r}) = -\nabla(\nabla \cdot \int_V G(r,r')\vec{F}(\vec{r'})\, d^3\vec{r'}) \tag{14}$$

While the rotational part is found to be

$$\vec{F_s}(r) = \nabla \times (\nabla \times \int_V G(r,r')\vec{F}(\vec{r'})\, d^3\vec{r'}) \tag{15}$$

It follows that

$$\vec{F(r)} = \vec{F_i}(\vec{r}) + \vec{F_s}(\vec{r}) \tag{16}$$

Zhou goes on to expand this conclusiuon further, but the proof of a solenoidal and an irrotational part are all that is required for the purposes of a brief summary.

IV. APPLICATIONS OF HELMHOLTZ'S THEOREM IN GAUGE TRANSFORMATIONS

One of the most important applications of Helmholtz's theorems lies in electromagnetic theory, and gauge transformations are an example of applied Helmholtz theory. Because the Helmholtz theorem is based on the division of a vector field into both a rotational and an irrotational part, parallels in accepted electromagnetic theory can be drawn for electric and magnetic fields. The potentials of the electric field $\vec{E}$ and the magnetic field $\vec{B}$ turn out to be decompositions, and the potentials are not unique. The magnetic vector potential is not defined uniquely because the divergence could be anything and still have no effect on the magnetic field; this leaves a degree of freedom. This degree of freedom allows for a choice in scalar and vector potentials, known as gauge.[4] Physical quantities do not depend on gauge, hence the term gauge invariance was coined to describe the degree of freedom that allows this choice of gauge parameters. The gauge can be chosen so that $\vec{E}$ and $\vec{B}$ are invariant, which is where the relation lies between gauge theory and Helmholtz's theorems.

By choosing a certain gauge one is able to determine physical properties that are difficult or impossible to obtain otherwise. Maxwells equations can be derived directly from gauge transformations and vice versa, depending on whether deductive or inductive methods are used. It can be shown that a decomposition of $\vec{E}$ by Helmholtzs theorem satisfies Maxwells equations, which suffice to determine the gauge. Some of the more common and useful gauge transformations include the Coulomb gauge and the Lorentz gauge.[5]

The Coulomb gauge is of particular interest because its charge density results in an instantaneous Coulomb potential which equals the scalar potential, and its properties are easily relatable to Maxwell's equations. For the Coulomb gauge,

$$\nabla \cdot \vec{A} = 0 \tag{17}$$

Applying Maxwell's equations, Poisson's equation emerges:

$$\nabla^2 \phi = -\frac{1}{\epsilon_0}\rho \tag{18}$$

To solve Poissons equation, one can simply set the potential to zero at infinity and solve. One advantage to using the Coulomb gauge is that the scalar potential is relatively easy to determine; this is because the Coulomb gauge allows for the decoupling of the scalar and vector potentials, which results in a wave equation for the vector potential that goes to zero outside the immediate vicinity of electric charges (Poissons equation is a general example of this). The main disadvantage, however, is that this decoupling also makes $\vec{A}$ particularly difficult to calculate.[6]

The Lorentz gauge is defined as

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \tag{19}$$

Which leads to two homogeneous wave equations for the potentials:

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\epsilon_0}\rho \tag{20}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \tag{21}$$

The solutions for the Lorentz gauge propagate at the speed of light c. The Lorentz gauge is the four-dimensional Poisson equation; it allows for multiple wave polarizations, which can correspond to classical radiation (transverse polarization). The main advantage in using the Lorentz gauge is that it allows for the use of the same differential operator (known as the dAlembertian) to determine both potentials, which is very useful for solving the inhomogeneous wave equation for specified sources. The main problem with the Lorentz gauge is that it sometimes requires additional parameters to reconcile its non-physical (i.e., non-transverse) components.[6]

V. DISCREPANCIES IN HELMHOLTZ'S THEOREM

The problem with the propositions of the Helmholtz Theorem is that they are too narrowly confined; these statements are not considered "rigorous and subjective", and there does not exist any obvious distinction between decomposition and its uniqueness. Second, the theorem is incomplete insomuch as it it limited to simply connected regions with single boundaries and does not allow for discontinuity, which is often experienced in practice as well as theory. Finally, these propositions show that Helmholtz's Theorem cannot be equivalent to a uniqueness theorem of a vector function.[2]

VI. CONCLUSIONS

In conclusion, it can be surmised the Helmholtz's Theorem is incomplete as it currently stands to explain complex physical properties, and is only valid in limited domains. While it is useful in the application of vector fields in simple problems, it fails to account for complexities often encountered in practice. Gauge transformations do allow for the extraction of physical data that is otherwise unaccessible, which is arguably the most useful application of the Helmholtz theorem. In general, Helmholtz's theorem is taken to be synonymous with the uniqueness theorem, which is not entirely true. By applying the propositions above, however, a uniqueness theorem for the Helmholtz equation itself emerges, and it completes the uniqueness theorem in dynamic theory of electromagnetics[1]. Unfortunately, while this applies to unbounded domain, there is no equivalence in the case of a bounded domain because the decomposition terms are not mutually orthogonal. In the future all areas will hopefully be able to offer uniqueness and completeness of decomposition on the case of bounded domain, how to build the difference and connection between Helmholtz theorem and vector field determinants, how to derive and define the irrotational field and solenoidal field components for every situation, and how to ascertain uniqueness.

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