QUIZ – 2, IIT HYDERABAD

ME5010 – Mathematical Methods for Engineers **Date: 13/08/2014**

Maximum Marks – 30 **Time duration: 1 hours**

Instructions:

i) Mobile is not allowed in the exam.
ii) Only calculator is allowed.

1. For a vector $\vec{A}$ of components (x_1, x_2, x_3) along the unit basis vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, write the components of $\vec{A}$
   (a) Along a transformed co-ordinates (q_1, q_2, q_3) having unit vectors $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ using *x*-*q* relationship.
   (b) In terms of the reciprocal basis, $(\hat{e}^1, \hat{e}^2, \hat{e}^3)$. Also write the $(\hat{e}^1, \hat{e}^2, \hat{e}^3)$ in terms of $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$.
   (c) Along the contravariant $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and covariant $(\hat{e}^1, \hat{e}^2, \hat{e}^3)$ systems, respectively.
   Also, obtain the relationship between contravariant and covariant components of vector $\vec{A}$ in terms of the components of metric tensor G. (Total Marks:$3 + 3 + 4 = 10$)

2. If $(\hat{e}_1, \hat{e}_2)$ is the basis vector of the Cartesian co-ordinate system P (x_1, x_2) then do the following
   (a) Find the unit vector $(\hat{e}_r, \hat{e}_\theta)$ in polar co-ordinate system (r, θ).
   (b) Find the Jacobian, J, of the transformation (x_1, x_2) to (r, θ).
   (c) Find the metric tensor G of the transformation.
   (d) Using G, prove that (r, θ) co-ordinate system is orthogonal and then obtain the metric coefficients(h, h_2).
   (e) Find the reciprocal basis $(\hat{e}^r, \hat{e}^\theta)$of the polar co-ordinate system (r, θ).
   (f) Obtain the co-variant and contravariant and co-variant components of the point P in terms of polar co-ordinates.
   (Total Marks: $2 + 2 + 2 + 2 + 2 + 4$=14)

3. For the co-ordinate transformation from (x_1, x_2) to (r, θ).
   (a)Write the differential arc length *ds* in terms of (dx_1, dx_2) and $(dr, d\theta)$, respectively, using the metric tensor G.
   (b) Using the formula $dA = \frac{\partial \vec{r}}{\partial q_1} \times \frac{\partial \vec{r}}{\partial q_2} dq_1 dq_2$, where $\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2$ and (q_1, q_2) is the transformed co-ordinate system, obtain the expressions of differential area element dA in Cartesian as well as polar co-ordinates, respectively.
   (Total Marks: $2 + 4 = 6$)

********* Good Luck**********

Solution - Quiz 2

Mathematical Methods

(1) e $\vec{A} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$

a) $(x_1, x_2, x_3) \longrightarrow (\hat{e}_1', \hat{e}_2', \hat{e}_3')$

$(x_1, x_2, x_3) \longrightarrow (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ ← (Given)

$\vec{e}_j' = \frac{\partial x_i}{\partial x_j}\hat{e}_i \Rightarrow$

$$\left.\begin{aligned} \hat{e}_1' &= \frac{\partial x_1}{\partial x_1}\hat{e}_1 + \frac{\partial x_2}{\partial x_1}\hat{e}_2 + \frac{\partial x_3}{\partial x_1}\hat{e}_3 \\ \hat{e}_2' &= \frac{\partial x_1}{\partial x_2}\hat{e}_1 + \frac{\partial x_2}{\partial x_2}\hat{e}_2 + \frac{\partial x_3}{\partial x_2}\hat{e}_3 \\ \hat{e}_3' &= \frac{\partial x_1}{\partial x_3}\hat{e}_1 + \frac{\partial x_2}{\partial x_3}\hat{e}_2 + \frac{\partial x_3}{\partial x_3}\hat{e}_3 \end{aligned}\right\} \text{Equation (A)}$$

$\vec{A} = a_1\hat{e}_1' + a_2\hat{e}_2' + a_3\hat{e}_3'$

$\vec{A} = (\vec{A}\cdot\hat{e}_1')\hat{e}_1' + (\vec{A}\cdot\hat{e}_2')\hat{e}_2' + (\vec{A}\cdot\hat{e}_3')\hat{e}_3'$

$\hat{e}_1', \hat{e}_2', \hat{e}_3'$ are given by eqn(A)

b) If $(\vec{e}^1, \vec{e}^2, \vec{e}^3)$ are reciprocal basis of vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$

$$\vec{A} = (\vec{A}\cdot\vec{e}^1)\hat{e}_1 + (\vec{A}\cdot\vec{e}^2)\hat{e}_2 + (\vec{A}\cdot\vec{e}^3)\hat{e}_3$$

$$\vec{e}^1 = \frac{\hat{e}_2\times\hat{e}_3}{\hat{e}_1\cdot(\hat{e}_2\times\hat{e}_3)} \;;\; \vec{e}^2 = \frac{\hat{e}_3\times\hat{e}_1}{\hat{e}_2\cdot(\hat{e}_3\times\hat{e}_1)} \;;\; \vec{e}^3 = \frac{\hat{e}_1\times\hat{e}_2}{\hat{e}_3\cdot\hat{e}_1\times\hat{e}_2}$$

such that $\vec{e}^1\cdot\hat{e}_1 = 1$; $\vec{e}^2\cdot\hat{e}_2 = 1$; $\vec{e}^3\cdot\hat{e}_3 = 1$;

c) Contravariant base vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and
Covariant base vectors $(\vec{e}_1', \vec{e}^2, \vec{e}^3)$

(i) $-\vec{A} = A^1\hat{e}_1 + A^2\hat{e}_2 + A^3\hat{e}_3 \Rightarrow (A^1, A^2, A^3)$ are contravariant components

(ii) $-\vec{A} = A_1\vec{e}_1' + A_2\vec{e}^2 + A_3\vec{e}^3 \Rightarrow (A_1, A_2, A_3)$ are covariant components

$\vec{A}\cdot e^1 = A^1\hat{e}_1'\cdot\hat{e}_1 + A^2\hat{e}_1'\cdot\hat{e}_2 + A^3\vec{e}^1\cdot\hat{e}_3$

$A^1 = \vec{A}\cdot e^1 \;;\; A^2 = \vec{A}\cdot e^2 \;;\; A^3 = \vec{A}\cdot e^3 \Rightarrow A^i = \vec{A}\cdot e^i$

$\vec{A} = A^i\hat{e}_i \Rightarrow \vec{A}\cdot e_j = A^i e_i\cdot e_j = A g_{ij}$ or $A_i = \vec{A}\cdot e_i$

$\vec{A} = A_i e^i \Rightarrow \vec{A}\cdot e^j = A_i\, e^i\cdot e^j = A_i g^{ij}$

$g_{ij} = e_i\cdot e_j$

$g^{ij} = e^i\cdot e^j$

$g^i_j = e^i\cdot e_j = \begin{cases} 1 & (i=j) \\ 0 & (i\neq j) \end{cases}$

$$\vec{A}\cdot e_j = \boxed{A_j = A^i g_{ij}} \qquad \boxed{A^j = A_i g^{ij}}$$

********* Good Luck**********

(2) $(\hat{e}_1, \hat{e}_2) \longrightarrow (\hat{e}_r, \hat{e}_\theta)$

$\vec{P}: (x_1, x_2) \longrightarrow (r, \theta)$

$\Downarrow$

$\boxed{x_1 = r\cos\theta \quad x_2 = r\sin\theta}$

a) Find unit vectors $(\hat{e}_r, \hat{e}_\theta)$

$\hat{e}_r = \frac{\partial x_1}{\partial r}\hat{e}_1 + \frac{\partial x_2}{\partial r}\hat{e}_2 = \boxed{\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2 = \hat{e}_r}$

$\hat{e}_\theta = \frac{\partial x_1}{\partial \theta}\hat{e}_1 + \frac{\partial x_2}{\partial \theta}\hat{e}_2 = \boxed{-r\sin\theta\,\hat{e}_1 + r\cos\theta\,\hat{e}_2 = \hat{e}_\theta}$

b) Jacobian $J = \frac{\partial(x_1, x_2)}{\partial(r, \theta)} = \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

$|J| = \det(J) = r(\cos^2\theta + \sin^2\theta) = r$

$\boxed{|J| = r}$

c) Metric tensor $G = J^T J = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -r\sin\theta\cos\theta + r\sin\theta\cos\theta \\ -r\sin\theta\cos\theta + r\sin\theta\cos\theta & r^2(\sin^2\theta + \cos^2\theta) \end{bmatrix}$

$G = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \Rightarrow g_{11} = 1 \quad g_{12} = 0 \quad g_{21} = 0 \quad g_{22} = r^2$

d) For the transformation from $(x_1, x_2) \to (r, \theta)$

Metric tensor $G = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$ $\because g_{12} = g_{21} = 0 \Rightarrow (r, \theta)$ is orthogonal

Metric coefficients $= h_1 = \sqrt{g_{11}} \;;\; h_2 = \sqrt{g_{22}}$

$\boxed{h_1 = 1 \;;\; h_2 = r}$

e) $\because \hat{e}_r = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2$

$\hat{e}_\theta = -r\sin\theta\,\hat{e}_1 + r\cos\theta\,\hat{e}_2$

$\hat{e}_z = \hat{e}_z$

$\hat{e}_\theta \times \hat{e}_z = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_z \\ -r\sin\theta & r\cos\theta & 1 \\ 0 & 0 & 1 \end{vmatrix}$

$(\hat{e}_\theta \times \hat{e}_z) = r(\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2) = r\hat{e}_r$

$\hat{e}_r \cdot (\hat{e}_\theta \times \hat{e}_z) = r$

$\Rightarrow$ reciprocal basis

$\hat{e}^r = \frac{\hat{e}_\theta \times \hat{e}_z}{\hat{e}_r \cdot (\hat{e}_\theta \times \hat{e}_z)} = \frac{\hat{e}_r}{\hat{e}_r \cdot \hat{e}_r} = \hat{e}_r$

For orthogonal system; $\hat{e}^r = \hat{e}_r \;;\; \hat{e}^\theta = \hat{e}_\theta$

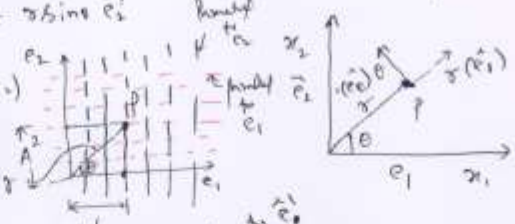
$\hat{e}^r = \frac{\hat{e}_\theta \times \hat{e}_z}{\hat{e}_r \cdot (\hat{e}_\theta \times \hat{e}_z)} = \frac{r\hat{e}_r}{r} = \hat{e}_r$

similarly, $\hat{e}^\theta = \hat{e}_\theta$

********* Good Luck**********

2 (f) Co-variant and contravariant component of point P in polar co-ordinate.

$\vec{P} = r\cos\theta\,\hat{e}_1 + r\sin\theta\,\hat{e}_2$



→ Contravariant component (e_1, e_2)

$A^1 = x_1 = r\cos\theta$

$A^2 = x_2 = r\sin\theta$

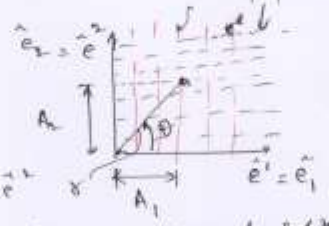
$\vec{P} = A^1 e_1 + A^2 e_2$

$\vec{P} = r\cos\theta\, e_1 + r\sin\theta\, e_2$

perpendicular to $\hat{e}^1$

perpendicular to $\hat{e}^2$

→ Covariant vector $(\hat{e}^1, \hat{e}^2)$



$\vec{P} = A_1\hat{e}^1 + A_2\hat{e}^2$

$\vec{P} = r\cos\theta\,\hat{e}^1 + r\sin\theta\,\hat{e}^2$

$\therefore \hat{e}^1 = \hat{e}_1$ & $\hat{e}^2 = \hat{e}_2$ (due to orthogonal system)

$\therefore |\vec{P}| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r = \sqrt{A_1A^1 + A_2A^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$

Here, $A_1 = A^1 = x_1 = r\cos\theta$

$A_2 = A^2 = x_2 = r\sin\theta$

3. $(x_1 - x_2) \longrightarrow (r, \theta)$

$\therefore x_1 = r\cos\theta$; $x_2 = r\sin\theta$

(a) $ds^2 = dx_i \cdot dx_i = dx_1^2 + dx_2^2 = dq_i g_{ij} dq_j$ $\because q_1 = r$; $q_2 = \theta$

$= dr\, g_{11} dr + dr\, g_{12} d\theta + d\theta\, g_{21} dr + d\theta\, g_{22} d\theta$

$\therefore G = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \Rightarrow \begin{matrix} g_{11} = 1 ; g_{12} = 0 \\ g_{21} = 0 ; g_{22} = r^2 \end{matrix} \Rightarrow \boxed{ds^2 = dr^2 + r^2 d\theta^2}$

(b) $dA = dx_1\, dx_2$ in Cartesian.

$= \left(\frac{\partial \vec{r}}{\partial q_1} \times \frac{\partial \vec{r}}{\partial q_2}\right) dq_1\, dq_2$ in polar (q_1, q_2) Co-ordinate system

If $q_1 = r$; $q_2 = \theta$ in polar co-ordinate.

$\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 = r\cos\theta\,\hat{e}_1 + r\sin\theta\,\hat{e}_2$

$\frac{\partial \vec{r}}{\partial q_1} = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2$; $\frac{\partial \vec{r}}{\partial q_2} = \frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\hat{e}_1 + r\cos\theta\,\hat{e}_2$

$\left(\frac{\partial \vec{r}}{\partial q_1} \times \frac{\partial \vec{r}}{\partial q_2}\right) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \hat{e}_3\left(r\cos^2\theta + r\sin^2\theta\right) = r\hat{e}_3$

$\therefore d\vec{A} = r\,dr\,d\theta\,\hat{e}_3 \Rightarrow \boxed{|dA| = dr\,r\,d\theta}$

********* Good Luck**********