Intersection Dimension and Maximum Degree

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Abstract We show that the intersection dimension of graphs with respect to several hereditary graph classes can be bounded as a function of the maximum degree. As an interesting special case, we show that the circular dimension of a graph with maximum degree Δ is at most $O(\Delta\frac{\log\Delta}{\log\log\Delta})$. We also obtain bounds in terms of treewidth.

Keywords: intersection dimension, graph coloring, hereditary classes.

1 Introduction

We consider finite, simple undirected graphs. A graph property or a graph class is a class of labeled graphs which are closed under isomorphism. A class P is a hereditary class if it includes all induced subgraphs of any of its members. In [4], Cozzens and Roberts termed a class P as dimensional if every arbitrary graph $G = (V, E)$ is the intersection of graphs from P, that is, there are k graphs $\{G_i = (V, E_i) \in P : 1 \leq i \leq k\}$ (for some k) such that $E = \cap_i E_i$. Kratochvil and Tuza, in their paper [3], showed that a class P is dimensional if and only if all complete graphs and all complete graphs minus an edge are in P.

Given a dimensional property $\mathcal{A}$, the minimum k such that a graph G can be written as the intersection of k graphs from $\mathcal{A}$ is defined as the *intersection dimension* of G with respect to $\mathcal{A}$ and is denoted by $dim_{\mathcal{A}}(G)$. Kratochvil and Tuza also proved that for any dimensional hereditary property P, either $dim_{\mathcal{A}}(G) = 1$ for every G or it can take arbitrarily large values. However, it may still be possible to express $dim_{\mathcal{A}}(G)$ in terms of other invariants of G.

Some interesting specializations of intersection dimension include the boxicity of a graph (the intersection dimension with respect to the class of interval graphs), cubicity (w.r.t. unit interval graphs), circular dimension (w.r.t. circular arc graphs), overlap dimension (w.r.t interval overlap graphs) and permutation dimension (w.r.t. permutation graphs). Of these, boxicity is the most well-known and various results on boxicity for special classes (like planar graphs, graphs of bounded treewidth) are known.

In this paper, we obtain bounds on the intersection dimension of a graph with respect to certain dimensional hereditary properties in terms of maximum degree. We also show that for such properties, the intersection dimension is bounded for graphs in any proper minor closed family and in particular, for planar graphs and for graphs of bounded treewidth. We also obtain significantly improved upper bounds for some special cases, notably the circular dimension (intersection dimension with respect to circular arc graphs) and permutation dimension. The proofs of these bounds are based on relating intersection dimension and forbidden subgraph colorings, in particular, frugal colorings.

The paper is organized as follows: Section 2 introduces some preliminaries we will require later. Section 3 presents some general results with respect to certain dimensional, hereditary classes which satisfy the the Zykov sum or FDC requirement (both defined in Section 2). In Section 4, we obtain an improved bound for the circular dimension. First, we need a few definitions and lemmas from [3].

2 Some Definitions and Lemmas

Definition 2.1 Following [3], we say that a class $\mathcal{A}$ of graphs satisfies the Full Degree Completion (FDC) requirement if for any graph $G = (V, E)$ in $\mathcal{A}$, the graph obtained by adding a new universal vertex (i.e. a vertex adjacent to all of V) is also in $\mathcal{A}$.

Definition 2.2 The Zykov sum of two graphs with disjoint vertex sets is formed by taking the union of the two graphs and adding all edges between the graphs. We say that a class $\mathcal{A}$ of graphs satisfies the Zykov Sum requirement if the Zykov sum of any two graphs in $\mathcal{A}$ is also in $\mathcal{A}$. It can be verified that if a hereditary class $\mathcal{A}$ satisfies the Zykov sum requirement, then $\mathcal{A}$ also satisfies the FDC requirement.

Lemma 2.2 ([3]) Let $\mathcal{A}$ be a class of graphs satisfying the Zykov sum requirement. If $G = (V, E)$ is a graph and $G_{ij} = (V_{ij}, E_{ij})$, $i = 1, 2, ...k$, $j = 1, \ldots, k_i$ are induced subgraphs of G such that (i) any each nonedge of G is present as a nonedge in some G_{ij} and (ii) for every i, the vertex sets $\{V_{ij}\}_j$ form a partition of V. Then, $dim_{\mathcal{A}}(G) \leq \sum_{i=1}^{k} \max\{dim_{\mathcal{A}} G_{ij}\}_j$.

Definition 2.3 For a family $\mathcal{F}$, we mean by $Forb(\mathcal{F})$ the set of all graphs which do not contain an isomorphic copy of any graph in $\mathcal{F}$ as a subgraph and by $\mathcal{G}(\mathcal{F})$ the set of all graphs which do not contain an isomorphic copy of any graph in $\mathcal{F}$ as an induced subgraph.

Following [2], we define a $(2, \mathcal{F})$-subgraph coloring of a graph G as a proper coloring of $V(G)$ such that the union of any 2 color classes is in $Forb(\mathcal{F})$. The minimum number of colors sufficient to obtain such a coloring is denoted by $\chi_{2,\mathcal{F}}(G)$. Well-known examples of such colorings include acyclic coloring ($\mathcal{F}$ is the set of even cycles), star coloring ($\mathcal{F}$ consists of P_4, the path on 4 vertices), β-frugal coloring ($\mathcal{F}$ is the star $K_{1,\beta+1}$).

Using Lemma 2.3 of [3] and Lemma 2.2 above, we obtain the following result which connects intersection dimension and $(2, \mathcal{F})$-subgraph colorings.

Theorem 2.4 Let $\mathcal{A}$ be a hereditary class of graphs which is closed under disjoint union and satisfying the FDC requirement. Let $\mathcal{F}$ be a family of connected bipartite graphs and suppose there exists a constant $t = t(\mathcal{F})$ such that for all graphs $H \in Forb(\mathcal{F})$, the intersection dimension of H with respect to the class $\mathcal{A}$ is at most t. Then, for any graph G, $dim_{\mathcal{A}}(G) \leq t\binom{\chi_{2,\mathcal{F}}(G)}{2}$. Further, if $\mathcal{A}$ satisfies the Zykov sum requirement, then $dim_{\mathcal{A}}(G) \leq t\chi_{2,\mathcal{F}}(G) + t$.

A non-trivial hereditary class of graphs which is closed under disjoint union and which satisfies the FDC requirement, must contain all stars. Therefore, by using the results of Albertson et al. on star coloring in the paper [1], we have the following corollary.

Corollary 2.5 Let $\mathcal{A}$ be a hereditary class of graphs which is closed under union and satisfying the Zykov sum requirement. Then for any graph G, we have $dim_{\mathcal{A}} \leq \chi_s(G)$ where $\chi_s(G)$ is the star chromatic number. Hence,

(i) $dim_{\mathcal{A}}(G) = O\left(\Delta^{3/2}\right)$ where $\Delta = \Delta(G)$;
(ii) if G has treewidth t, $dim_{\mathcal{A}}(G) \leq t(t-1)/2$.

If we only know that the $\mathcal{A}$ satisfies FDC, then
(iii) $dim_{\mathcal{A}}(G) = O\left(\Delta^3\right)$ where $\Delta = \Delta(G)$;
(iv) if G has treewidth t, $dim_{\mathcal{A}}(G) = O(t^4)$.

Finally for graphs of genus g, the star chromatic number is known to be atmost $O(g)$ so that the intersection dimension w.r.t FDC (resp. Zykov sum) satisfying classes, for such graphs is atmost $O(g^2)$ (resp. $O(g)$). More generally, it is known from the results of [1] that for any proper minor closed family $\mathcal{C}$, $\max_{G \in \mathcal{C}} \chi_s(G)$ is bounded by a constant and hence $\max_{G \in \mathcal{C}} dim_{\mathcal{A}}(G)$ is also bounded by a constant, whenever $\mathcal{A}$ is closed under union and satisfies the FDC requirement.

3 Improved bounds

In Theorem 3.2 (stated below), we considerably improve the bounds in Corollary 2.5 by combining Theorem 2.4 with the following result (Theorem 3.1) of Molloy and Reed [5] on frugal coloring.

Theorem 3.1 ([5]) Let G be any graph of maximum degree Δ. Then G can be properly colored using $\Delta + 1$ colors so that any vertex has at most β neighbors in any color class, where $\beta = O((\log \Delta)/(\log \log \Delta))$.

Theorem 3.2 Suppose that for a hereditary class $\mathcal{A}$ which is closed under union and also satisfies the Zykov sum requirement, the following can be shown : For any graph G of maximum degree Δ, $dim_{\mathcal{A}}(G) \leq c\Delta^t$ where c and t are positive constants. Then in fact, the following holds:

(i) For any graph G, $\dim_{\mathcal{A}}(G) \leq \Delta(\log \Delta)B^{(\log^* \Delta)}$ for some constant B.
(ii) If $\mathcal{A}$ satisfies the FDC requirement but not necessarily the Zykov sum requirement, then $dim_{\mathcal{A}}(G) \leq \Delta^2(\log \Delta)B^{(\log^* \Delta)}$ for some constant B.
(iii) In particular, if $\mathcal{A}$ is the class of all permutation graphs, then for any G, $dim_{\mathcal{A}}(G) \leq \Delta(\log \Delta)B^{(\log^* \Delta)}$.

The assumption of closure under union used in Theorem 3.2 is essential, as otherwise the dimension number need not always be expressed as a function of the maximum degree as the following examples illustrate.

Unbounded dimension number with only FDC assumption: Consider the hereditary class of graphs consisting of cliques and cliques minus edges. This is the intersection of all dimensional classes satisfying FDC. The intersection dimension of a graph G w.r.t. this class is $|E(G^c)|$ which is not bounded by any function of the maximum degree of G.

Unbounded dimension with Zykov Sum assumption: The Zykov sum assumption carries over intersection and thus we can consider the smallest hereditary and dimensional class of graphs satisfying the ZS assumption. This smallest class is in fact the set of all cliques plus cliques minus a matching (of any size). It is easy to see that the intersection dimension of a graph G w.r.t this class is in fact $\chi'(G^c)$. This shows that for classes obeying the ZS assumption too, the intersection dimension need not always be bounded by a function of the maximum degree.

4 Circular dimension - A Special Case

Circular arc (CA) graphs are defined as the intersection graphs of arcs of a circle. Despite their similarity to interval graphs (which are a subclass of CA graphs), these need not be perfect graphs while interval graphs are also perfect graphs. Also, no complete forbidden induced subgraph characterization is known for the class CA. The corresponding intersection dimension is known as the CA-dimension and is denoted by $dim_{CA}(G)$.

Since CA is a super class of interval graphs, it follows that for any G, $dim_{CA}(G) \leq boxicity(G)$. However, while a tight upper bound on the boxicity of an arbitrary graph is still unknown (conjectured to be $O(\Delta)$), we shall show that $dim_{CA}(G)$ is close to a linear function of Δ.

Lemma 4.1 Let G be a split graph such that every clique vertex has at most t neighbors in the independent set. Then $dim_{CA}(G) \leq t + 1$.

Proof of Lemma 4.1 Form $t + 1$ CA graphs $G_0, G_1, ..., G_t$ with $G = G_0 \cap G_1 \cap ... \cap G_t$ as follows. Assume, w.l.o.g., that $I = \{1, \ldots, n\}$ constitute the independent set in G. Consider $n + 1$ equally distanced and distinct points on the unit circle and label them consecutively with $0, 1, \ldots, n$ as you traverse in the clockwise direction. In each G_r, each $i \in I$ is identified with the circular arc consisting just i. For any clique vertex u with r neighbors $i_1 < i_2 < \ldots < i_r$ and for any $s, 0 \leq s \leq r$, we identify u with the circular arc (in the clockwise direction) joining $s + 1$ with s (the addition being modulo $r + 1$) in the graph G_s. For $s > r$, we identify u (in G_s) with the circular arc used in G_r. It can be verified that $E(G) = E(G_0) \cap \ldots \cap E(G_t)$ and each G_i is in CA. This proves the lemma.

Theorem 4.2 The circular dimension of any graph G of maximum degree Δ satisfies: $dim_{CA}(G) = O(\Delta \frac{\log \Delta}{\log \log \Delta})$.

Proof of Theorem 4.2 Using Theorem 3.1, we obtain a $\beta = O\left(\frac{\log \Delta}{\log \log \Delta}\right)$-frugal coloring $(V_1, \ldots, V_k)$ of $V(G)$ using $k = \Delta + 1$ colors. We now form k split supergraphs $G_1, \ldots, G_k$ where G_i is obtained from G by making $G[V - V_i]$ a complete graph. It can be seen that $E(G) = E(G_1) \cap \ldots \cap E(G_k)$. Now we apply Lemma 4.1 to each G_i and deduce that $dim_{CA}(G_i) \leq \beta + 1$ and hence $dim_{CA}(G) \leq k(\beta + 1) = O(\Delta \frac{\log \Delta}{\log \log \Delta})$. This proves the theorem.

In this context, we recall that Shearer [6] has shown that there exist graphs on n vertices for which the circular dimension is at least $\Omega(\frac{n}{\log_2 n})$.

Conclusions : For some hereditary classes with a single forbidden induced subgraph H, we have obtained bounds on the intersection dimension the details of which are skipped due to lack of space. An open problem is to narrow the gap between the upper and lower bounds for circular dimension.

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