# Assignment I

CS1210: Proof Techniques

**Instructions:**

**Submit the assignment at or before the beginning of the class on 09.09.2014.**

1. Which of the following statements are true for a real number $x$?

   (a) If $x = 3$, then $x^2 = 9$. (b) If $x^2 = 9$, then $x = 3$.

   (c) If $x^2 \neq 9$, then $x \neq 3$. (d) If $x \neq 3$, then $x^2 \neq 9$.

2. For each proposition below, decide whether it is true or false and give a brief explanation.

   (i) $\forall x \in \mathbb{Z} : ((x < 0) \vee (x^2 \geq x))$.

   (ii) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : (x = y + 1)$.

   (iii) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z} : (x = y + 1)$

   (iv) $\forall p, q \in \{T, F\} : (p \vee q) \rightarrow ((p \wedge \neg q) \vee (\neg p \wedge q))$.

   (v) $\forall x, y \in \mathbb{R} : (((x = 5) \wedge (y = 1)) \rightarrow ((x > 10) \vee (y > 0)))$.

3. The drinker's paradox says that in a room with a non-empty set of people, there is a person $P$ such that if $P$ is drinking, then everyone in the room is drinking. Formally, if $S \neq \emptyset$, then $\exists p \in S : (p \text{ drinks} \rightarrow (\forall q \in S : q \text{ drinks}))$. Prove this!

4. Three friends Pablo, Dali, and Henri are talking to each other about the art collection of Remo. Pablo says: "Remo has at least four paintings of Vinci." Dali says: "No, he has less than four paintings of Vinci." "According to me," says Henri, "Remo has at least one Vinci." You know that exactly one of the three friends is correct. How many paintings of Vinci does Remo have?

5. We define the *least common multiple* or *lcm* of two non-zero integers as: $lcm(a, b) = min\{L \in \mathbb{N} : a|L \text{ and } b|L\}$. We also define $lcm(a, 0) = lcm(0, a) = 0$ for all $a \in \mathbb{Z}$.

For example, $lcm(18,-21) = lcm(18,21) = 3$. Prove: If $a, b, c$ are integers, then $lcm(ca, cb) = c \cdot lcm(a, b)$.

[Hint: Show that L.H.S. $\leq$ R.H.S. and R.H.S. $\leq$ L.H.S.]

6. Let $S$ be a set of four real numbers whose product is equal to 16.

   Show that (i) $S$ contains two numbers whose product is at most 4; (ii) $S$ contains three numbers whose product is at most 8.

7. Show that there do not exist natural numbers $a, b, c$ such that $4a^3 + 2b^3 = c^3$.

   [Hint: As a first step, show that $c$ is even. Later use the well-ordering property of $\mathbb{N}$ - consider for example, the set $\{a \in \mathbb{N} : \exists b, c \in \mathbb{N} : 4a^3 + 2b^3 = c^3\}$.]

8. Let $f$ and $g$ be differentiable functions such that $f(x)g(x) = x$. Show that $f(0) \neq 0$ or $g(0) \neq 0$.