Discrete Mathematics: Practice Problems

1. For the two statements below, decide whether they are true or false.

   (i) $\exists n \in \mathbb{N} : \forall m \in \mathbb{N}, ((m < n) \to (m + n \text{ is even}))$.

   (ii) $\forall n \in \mathbb{N}, \exists m \in \mathbb{N} : (m \geq n) \wedge (\exists r \in \mathbb{N} : m = r^2)$.

2. Prove that $\log_2 3$ is irrational.

3. Prove: If $x \in \mathbb{R}$, then $x(4 - x) \leq 4$.

4. Prove that $4^n + 6n - 1$ is divisible by 9 for any natural number $n \geq 0$.

5. Prove that $3^n \geq 2^n + 5n$ for all $n \geq 2$.

6. Given a set S of 11 integers, show that there must be two different elements $x, y \in S$ such that $x - y$ is divisible by 10.

7. The numbers $1, 2, \ldots, 9$ are divided into 3 groups. Show that the product of the numbers in at least one group must be greater than 71.

8. Let S be any set and let $f : S \to S$ be a function such that $f(f(x)) = x$ for all $x \in S$. Show that f is a bijection.

9. Find the remainder when 7^{21} is divided by 43.

10. How many 6-digit natural numbers have all their numbers of the same parity (all even or all odd)?

11. In how many ways can you pick three different numbers from $\{1, 2, \ldots, 3n\}$ so that the sum of the three numbers is divisible by 3?

12. Let $f(n)$ be the number of n-letter strings formed using the letters in $\{A, B, C\}$ and containing neither two consecutive As nor two consecutive Bs. Find a recurrence relation for $f(n)$.

13. Given a string $X_1X_2 \ldots X_{2n}$, consisting of n As and n Bs, we say that it is balanced if for every $k \leq 2n$, the number of As in the substring $X_1X_2 \ldots X_k$ is greater than or equal to the number of Bs in that substring. For example, the string $AABABB$ is balanced while $ABBAAB$ is not balanced. Let $C(n)$ denote the number of balanced strings of length $2n$. Find a recurrence relation for $C(n)$.

14. Let $D(n)$ be the number of derangements of $\{1, 2, \ldots, n\}$. Give a combinatorial proof that $D(n) = (n-1)\,(D(n-1) + D(n-2))$.

15. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ and let $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Find a recurrence relation for a_n.

16. For each pair f, g of functions below, decide which of the four statements (A) $f(n) = O(g(n))$, (B) $f(n) = o(g(n))$, (C) $g(n) = O(f(n))$, (D) $g(n) = o(f(n))$ is/are true. Note that more than one statement might be true, for example, (B) always implies (A).

    (i) $f(n) = n^6$, $g(n) = 10n^4$

    (ii) $f(n) = 5n^2 + 1$, $g(n) = n^2 \log n$

    (iii) $f(n) = n^{\log n}$, $g(n) = (\log n)^n$

    (iv) $f(n) = sin^2(n\pi)$, $g(n) = cos^2(n\pi)$

17. Let $F : \mathbb{N} \to \mathbb{N}$ be a function satisfying: $F(n) \leq 2F(n/3) + n$ for all n. Show that $F(n) = O(n)$.