Assignment II

CS1010: Discrete Mathematics for Computer Science

Instructions:

Submit the assignment at or before the beginning of the class on 27.01.2014.

You are encouraged to think about the bonus problems as well.

1. Prove by mathematical induction: If n is an integer greater than 1, then the following holds:
$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

2. The Fibonacci sequence $F_1, F_2, F_3, \ldots$ is defined by the values $F_1 = 1, F_2 = 1$ and the relation $F_n = F_{n-1} + F_{n-2}\ \forall n \geq 3$. The first seven values in this sequence are $1, 1, 2, 3, 5, 8, 13$. Prove by induction that F_n is even if and only if n is divisible by 3.

3. Consider a round-robin tournament involving $n \geq 2$ teams, i.e. every team plays every other team exactly once. Show that after all the games are over, we can assign each team a label from $\{1, 2, \ldots, n\}$ such that the following two conditions hold:

   (i) No two teams get the same label.

   (ii) $\forall i \in \{1, 2, \ldots, n-1\}$, the team labeled i won against the team labeled $i+1$.

4. For $n \geq 1$, let $H(n) = \sum_{i=1}^{n} \frac{1}{i}$. Prove the following.

   (i) For every natural number k, the inequality $H(2^k - 1) \leq k$ holds.

   (ii) For every natural number n, there exists an integer $k \geq 1$ such that $2^{k-1} \leq n < 2^k$.

   (iii) For every natural number n, the following inequality holds: $H(n) \leq 1 + \log_2 n$.

5. Rahul left his cycle in a parking lot that is filled with bikes and cycles, parked neatly in several rows. When he returns, he notices someone push all the vehicles on the left end of each row, causing them to fall. Rahul knows that if a bike falls, then the

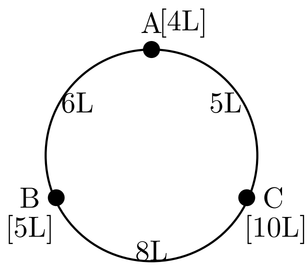
vehicle to its immediate right will also fall, but if a cycle falls, then the next vehicle will fall only if that is also a cycle. Rahul hasn't spotted his cycle yet, but he deduces by mathematical induction that it will fall. Here is his proof:

"Let $P(n)$ be the statement: 'In every row, every cycle in nth position will fall.' I know that $P(1)$ is true - all cycles in the first position were pushed. For the inductive step, consider any row with a cycle in the nth position. By the inductive assumption, this cycle will fall. Now consider the next vehicle, which is in the $(n+1)$ th position. If it is a bike, I don't have to prove anything; if it is a cycle, it will fall because it was pushed by the cycle on its left. Thus by induction $P(n)$ is true for all n." What is the flaw in his argument?

Bonus problems:

6. The following is one variation of the game of Nim. A number of sticks is placed on the ground and two players take turns removing the sticks. In each turn, a player may remove one, two or three sticks. The last player to remove a stick loses. Prove that if the number of sticks is of the form $4k+1$, where $k \in \mathbb{N}$, then the second player has a strategy to win.

7. There are n cars on a circular track. Together they have enough fuel to complete one lap (round) of the track. Show that there is a car which can complete one lap by collecting fuel from the other cars on its way round the track. An example is shown below. The number on an arc indicate the amount of petrol needed to travel that arc. Neither Car A nor Car B can complete a lap, but Car C can complete one lap by collecting petrol from car B, then car A and finally returning to its starting position.



A note on the induction step

Consider the following ways of using mathematical induction. Let $P_0, P_1, P_2, \ldots$ be a sequence of propositions.

Example 1:

(a) If P_0, P_1 are true and $P_n \Rightarrow P_{n+2}$, then $P(n)$ is true for all $n \geq 0$.

(b) If P_0, P_1, P_2, P_3, P_4 are true and $P_n \Rightarrow P_{n+5}$, then P_n is true for all $n \geq 0$.

(c) If for some $k \geq 1$, $P_0, P_1, \ldots, P_{k-1}$ are true and $P_n \Rightarrow P_{n+k}$, then P_n is true for all $n \geq 0$.

Example 2:

(a) If P_0, P_1 are true and $P_n \Rightarrow P_{2n}$ and $P_n \Rightarrow P_{2n+1}$, then P_n is true for all $n \geq 0$.

(b) If P_0, P_1, P_2, P_3, P_4 are true and $P_n \Rightarrow P_{5n}$, $P_n \Rightarrow P_{5n+1}$, $P_n \Rightarrow P_{5n+2}$, $P_n \Rightarrow P_{5n+3}$, $P_n \Rightarrow P_{5n+4}$, then P_n is true for all $n \geq 0$.

(c) If for some $k \geq 1$, $P_0, \ldots, P_{k-1}$ are true and $P_n \Rightarrow P_{kn}$, $P_n \Rightarrow P_{kn+1}$, $\ldots$, $P_n \Rightarrow P_{kn+k-1}$, then P_n is true for all $n \geq 0$.

Both the examples are valid induction proofs, but proving the induction step through statements such as $P_n \Rightarrow P(f(n))$ where $f(n) > n$, requires a great degree of care (see problem 5!). The two examples prove different statements (P_{n+5}, $P_{2n} \wedge P_{2n+1}$ etc) and the second example involves proving not one but several statements in the induction step. The key point is:

The induction step is a deduction.

The goal of the induction step is to deduce just one statement, namely $P(n)$, for an arbitrary value of n not proved as a base case. Thus, it is a good idea to stick to the following classical model or ensure that your inductive proof can be reduced to it.

Example 3: If $P_0, \ldots, P_{k-1}$ are true and $(P_0 \wedge \ldots \wedge P_{n-1}) \Rightarrow P_n$ for all $n \geq k$, then P_n is true for all values of n.

Examples 1 and 2 are really special cases of Example 3. For instance, example 2(a) is equivalent to deducing P_n from $P_{\lfloor n/2 \rfloor}$ for all $n \geq 2$.