Assignment I

CS1010: Discrete Mathematics for Computer Science

Instructions:

Submit the assignment at or before the beginning of the class on 20.01.2014.

The bonus problems are optional but you are encouraged to think about them.

1. For each proposition below, decide whether it is true or false and give a brief explanation. Assume the universe (domain of variables) to be $\mathbb{Z}$, the set of integers.

   (i) $((x = 5) \wedge (y = 1)) \rightarrow ((x > 10) \vee (y > 0))$

   (ii) $\forall x((x < 0) \vee (x^2 \geq x))$ (iii) $\forall x \exists y(x < y)$

   (iv) $\exists x \forall y(x < y)$ (v) $\neg(\exists x P(x)) \leftrightarrow (\forall x \neg P(x))$ for all predicates $P(x)$

2. The drinker's paradox says that in a room with a non-empty set of people, there is a person P such that if P is drinking, then everyone in the room is drinking. Formally, if $S \neq \emptyset$, then $\exists p \in S : (p \text{ drinks} \rightarrow (\forall q \in S : q \text{ drinks}))$. Prove this!

3. Given a real number x, the greatest integer which is less than or equal to x is denoted by $\lfloor x \rfloor$ (read as floor of x). For example $\lfloor 3.24 \rfloor = 3$, $\lfloor 3 \rfloor = 3$ and $\lfloor -5.27 \rfloor = -6$.

   Prove the following: If n is a natural number with binary representation $a_r a_{r-1} \ldots a_1 a_0$, then for any $j \in \{0, 1, \ldots, r\}$, the binary representation of $\lfloor \frac{n}{2^j} \rfloor$ is $a_r a_{r-1} \ldots a_j$.

4. Prove that there does not exist an integer $n > 3$ such that $n, n+2, n+4$ are all prime numbers.

5. Let $n \geq 2$ and let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Prove that there is some $i \in \{1, 2, \ldots, n\}$ such that $a_i \geq \frac{1}{n}$.

6. Prove that if n is a natural number, then $\sqrt{n}$ is rational if and only if n is a perfect square. Hint: First show that if n is not a perfect square, then there is a prime p which divides n to an odd power - you may use the following result, the proof of which we shall see later.

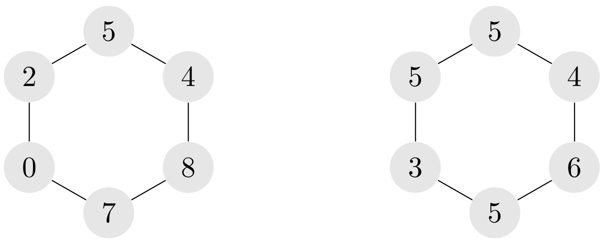
Theorem (Unique Factorization): Every natural number can be written as the product of primes in a unique way.

For example, the unique prime decomposition of the number 60 is $60 = 2^2 \times 3^1 \times 5^1$. The prime factorization of an arbitrary number n will look like: $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, where the p_is are the primes dividing n.

Bonus problems:

7. A simple test for divisibility for 3 for a natural number n uses the observation that n is divisible by 3 if and only if the sum of the digits of n is divisible by 3. Can you think of a divisibility test by 3 when n is given in binary?

8. Consider 6 numbers placed on the vertices of a hexagon, as in the figure (a) below.



(a) (b)

Suppose that you are allowed to increase or decrease any two adjacent numbers by the same value. For example, (b) was obtained from (a) by two such operations.

Can you make all the numbers equal by a sequence of such operations?

9. [Exercise:] Read the proof of the solution to Sam Loyd's 15-puzzle from the Lehman-Leighton lecture notes (2004 version).

10. The Babylonian method (later generalized by Newton) to find square roots works like this: If Y is a given positive real number, we define X_0 to be some positive real number (an initial guess for $\sqrt{Y}$). Then for $i \geq 1$, X_{i+1} is computed as $X_{i+1} = \frac{1}{2}(X_i + \frac{Y}{X_i})$.

Prove that for any starting value and for any $i \geq 1$, (i) $X_i \geq \sqrt{Y}$ and (ii) $X_{i+1} \leq X_i$ (hence X_{i+1} is closer to $\sqrt{Y}$ than X_i).

Further, suppose that X_0 lies in the interval $[0, Y]$. What can you say about the number of steps needed to achieve an error less than 1%?