Assignment V

CS1010: Discrete Mathematics for Computer Science

Instructions:

Submit the assignment by the beginning of the class on 14.04.2014. The program may be submitted by email.

1. Find the remainder when n is divided by q for the following (n, q) pairs:

   (i) $n = \sum_{k=1}^{24} k^{44}$, $q = 43$;

   (ii) $n = 7^{2048}$, $q = 60$;

   (iii) $n = a^{(b^c)}$ where $a = 11$, $b = 111$, $c = 1111$, $q = 7$.

2. Find all integers x satisfying all the following relations: $x = 2 (\text{mod } 3)$, $x = 3 (\text{mod } 5)$, $x = 5 (\text{mod } 7)$, $x = 7 (\text{mod } 11)$.

3. Write a C program that accepts two positive integers m and n and finds (i) gcd (m, n), (ii) $x, y \in \mathbb{Z}$ such that $mx + ny = \gcd(m, n)$.

   Solve **ANY TWO** of the remaining problems.

4. In class, we proved that any planar graph on $n \geq 4$ vertices has at most $3n - 6$ edges. The proof began by assuming that the graph is connected and used Euler's polyhedral formula together with the fact that every face has at least 3 edges.

   Prove that any planar graph on $n \geq 5$ vertices and not containing a triangle has at most $2n - 4$ edges. [Hint: Every face must have at least 4 edges.]

   Deduce the solution for the utilities puzzle.

5. Consider the following algorithm:

   Input: Two positive integers a, b

```
Algorithm:
Set x=a, y=b, u=a, v=b.
while (x is not equal to y)
{
    if (x>y)
      Set x=x-y, u=u+v.
    else
      Set y=y-x, v=v+u.
}
Output x, (u+v)/2.
```

We know that the value of x that is output is equal to $gcd(a, b)$. Show that the other value that is output is equal to $lcm(a, b)$.

[Hint: Find a loop invariant involving x, y, u, v and then use the following relation: $gcd(a, b)lcm(a, b) = ab$.]

6. Let p be a prime. For $a \in \{1, 2, \ldots, p-1\}$, we defined a^{-1} as the unique $x \in \{1, 2, \ldots, p-1\}$ such that $ax = 1(\text{mod } p)$. Show that:

   (i) $(a^{-1})^{-1} = a$;

   (ii) $a = a^{-1}$ if and only if $a = 1$ or $a = p - 1$.

   Deduce that $(p-1)! = -1(\text{mod } p)$. This result is known as Wilson's theorem.

7. A sequence of 100 coins is placed in a row on a table so that the faces of the coins display Heads and Tails in alternation (i.e. the sequence is $H, T, H, T, \ldots H, T$). Suppose that we wish to flip some of the coins so that all the coins are Tails up. The simplest way to do this is to flip the 50 coins that display Heads. This requires 50 flips. Now suppose that in a single "flip", you are allowed to turn over any sequence of consecutive coins. Can you reduce the number of flips to less than 50? Justify your answer.

8. Given a connected graph $G = (V, E)$, an Eulerian trail for G is a sequence $v_1, v_2, \ldots, v_k$ of vertices such that $\{v_i, v_{i+1}\} \in E$ for all $i < k$ and such that every edge is present exactly once in the trail (i.e. for each edge $\{u, v\}$ there is a unique i such that $\{v_i, v_{i+1}\} = \{u, v\}$.

   The goal of this exercise is to prove the "other direction" of Euler's theorem: **Let G be a connected graph. If the number of vertices in G of odd degree is zero, then for every vertex v, there is an Euler trail that starts (and ends) at v.**

If there are exactly two vertices of odd degree (say u and v), then there is an Euler trail that starts at u (and by symmetry, one that starts at v).

A natural idea is to prove this by induction on the number of edges. Notice that as soon as one edge (say e) is traversed, the problem reduces to finding an Euler trail in the graph obtained by deleting e, which has a fewer number of edges (and hence induction can be applied).

For the induction step, it is easy to see that the parity condition is preserved for the reduced graph, so the only problem that can potentially occur is that the edge traversed in the first step is a bridge. Problem 6 of the previous assignment ensures that this cannot happen when all the degrees are even; the other case is left to you.

Write a proof of Euler's theorem using induction on the number of edges, based on the ideas sketched above.