

Problem Set I

CS1250: Combinatorics

Instructions:

You are encouraged to attempt most of the problems before the next class.

Problems:

1. Let $S = \{1, 2, \dots, 10^8\}$.
 - (i) How many numbers in S contain the digit 1?
 - (ii) How many numbers in S contain the digit 1 or the digit 7?
 - (iii) How many numbers in S contain the digit 1 or the digit 7 but not both?
2. Let d, n be natural numbers. Then d is said to be a divisor of n if $\frac{n}{d} \in \mathbb{N}$.
Let $N = 2^{10} \times 3^8 \times 5^{11}$.
 - (i) How many divisors does the number N have?
 - (ii) What is the sum of all the divisors of N ?
3. Find the number of subsets of $\{1, 2, \dots, n\}$ whose size is even.
4. In how many ways can the letters of NEWYORK be arranged without containing NEW or YORK as substrings?
5. Find the number of ways to partition mn people into m groups of n each.
6. Let $m, n \geq 1$ be natural numbers. Suppose that you are at $(0, 0)$ on the plane and at each step you can increase either your x -co-ordinate or y -co-ordinate by 1.
 - (i) In how many ways can you reach the point (m, n) ?

- (ii) In how many ways can you reach the point (m, n) without going through (r, s) , where $0 < r < m$, $0 < s < n$?
7. We have three boxes labeled A,B,C and 100 distinct books to be put inside these boxes. In how many ways can we do this if the number of books in box A is to be 30, the number of books in box B is to be 50 and the number in box C is to be 20?
8. Find the number of subsets S of $\{1, 2, 3, 4, 5, 6, 7\}$ such that the sum of the elements of S is divisible by 7.
9. A standard deck of playing cards contains 52 cards - four suits of 13 cards each. The suits are spades (\spadesuit), clubs (\clubsuit), hearts (\heartsuit) and diamonds(\diamondsuit). The 13 cards in each suit are labeled A,2,3,...,10,J,Q,K.]
- A set of 5 cards is said to be a *hand*.
- (i) How many hands form a flush, i.e. all five cards being from the same suit?
- (ii) How many hands contain Four Of a Kind, i.e. four cards with the same value, four eg: four As, four 7s etc.?
- (iii) How many hands form a Straight, i.e. 5 cards in increasing order such that not all are from the same suit? The lowest card is A although a Straight can also end in an A. [Eg: $A\clubsuit, 2\spadesuit, 3\spadesuit, 4\heartsuit, 5\diamondsuit$ and $10\diamondsuit, J\spadesuit, Q\heartsuit, K\heartsuit, A\spadesuit$]
10. Consider the iterated substitution cipher which works as follows:
- Given a permutation π of A, B, \dots, Z , the i th letter of the message is substituted i times using π .
- For example, if the permutation were $A \rightarrow Z, B \rightarrow A, C \rightarrow B, \dots, Z \rightarrow Y$, and the message is *CODE*, then the ciphertext would be *BMAA*.
- For an unknown key π , the following two plaintext-ciphertext values are known.
- Plain-text 1: "THEQUICKBROWNFOXJUMPSOVERTHELAZYDOG"
- Cipher-text 1: "XXIIGQYKJRCWRRCLKKPAYFQRZZIPAXYNYU"
- Plain-text 2: "PERMUTATION"
- Cipher-text 2: "BAVMGZQTAYV"
- Find the key and decipher the ciphertext "TAJAWRCVASIREDISXOCFJAVYP".