Problems on Probability and Computing

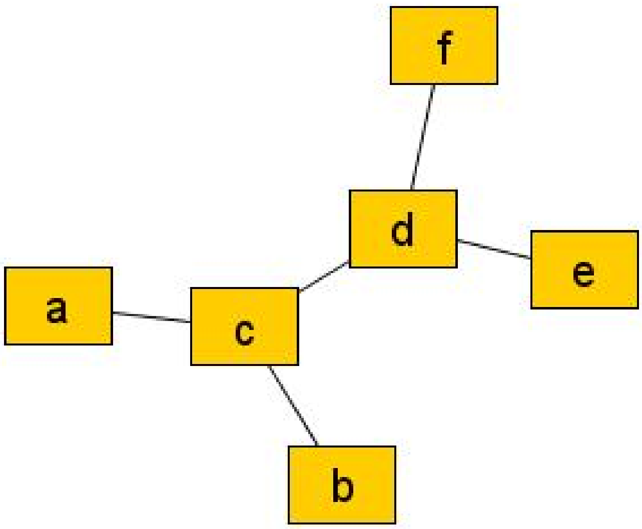
1 Exercises

1. Let A be a randomized Monte Carlo algorithm that solves a decision problem in polynomial time, and which is correct with probability at least $\frac{1}{n^3}$, where n is the size of the input, and the probability is over the random choices made by the algorithm. Describe a polynomial time algorithm to solve the same problem whose probability of correctness is at least $\frac{99}{100}$.

2. For the graph below and for each of the following cuts, find the probability that Karger's algorithm outputs that cut.

   (i) $\{a, b, c\}$ and $\{d, e, f\}$

   (ii) $\{a\}$ and $\{b, c, d, e, f\}$



3. Let X, Y be the values on two fair 6-sided dice. Find (i) $E[X|X + Y \geq 11]$ and (ii) $E[X + Y|X - Y = 1]$.

4. Fifty numbers are picked u.a.r and independently from $\{1, 2, 3, \ldots, 100\}$. Let $X =$ number of distinct elements picked. Find $E[X]$.

5. Let X be picked u.a.r from $\{1, 2, \ldots, 3000\}$. In each of the following two cases, decide with justification whether the pair A, B of events are independent.

   (i) A: X is divisible by 2. B: X is divisible by 3.

   (ii) A: X is divisible by 200. B: X is divisible by 300.

6. Consider a set of m clauses over n variables, where each clause is an OR of three literals. An example is $\{(x \vee \bar{y}\bar{z}), (\bar{x} \vee z \vee \bar{w})\}$ with $m = 2$ clauses, over $n = 4$ variables. For a random assignment of True/False to each variable, calculate the expected number of clauses that are made true by the assignment.

7. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a convex function and $S \geq 0$. We want to find $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \ldots + x_n = S$ and $\sum_{i=1}^{n} f(x_i)$ is minimized. Show that the minimum is attained when all the x_is are equal to S/n.

8. A fixed point of a permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a value i such that $\pi(i) = i$. Find the variance in the number of fixed points of a permutation chosen uniformly at random from all permutations. (Hint: Let $X_i = 1$ if i is a fixed point and $X_i = 0$ otherwise. Find $E[X_i X_j]$.)

2 Problems

9. An array A initially has $A[i] = ai + b$ (modulo n) for $i = 0, 1, \ldots, n-1$, where $gcd(a, n) = 1$. Subsequently the array becomes corrupt: up to 1/5th of the entries are replaced by garbage values. Let's call the corrupt array B to which you have access. Given an index i, your goal is to find $A[i]$. Design a simple randomized algorithm for this.

   Hint: If you know that $a = 1$, how would you solve it?

10. Suppose that you see a stream of distinct numbers. When you see the first number, you store its value, and subsequently, when you see the ith number, you do the following: with probability $\frac{1}{i}$, you replace the current number stored with the new number. Show that this method maintains numbers from the stream with uniform distribution, that is: after seeing i numbers, the probability of maintaining each of the numbers seen so far is $\frac{1}{i}$.

11. Let A be an algorithm that answers a decision problem (True/False) correctly with probability $\frac{1}{2} + \alpha$, for some $\alpha > 0$. To solve the decision problem with a higher probability of correctness, suppose that we repeat the algorithm $2k + 1$ times, and choose the decision that occurs more than k times. Using Chebyshev's inequality, find

a value of k (in terms of α), which ensures that the above "majority voting" algorithm is correct with probability at least 0.9.

12. Since $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$, we have $P(A \cup B) \leq P(A) + P(B)$. More generally, we have $Pr(A_1 \cup \ldots \cup A_n) \leq Pr(A_1) + \ldots + Pr(A_n)$. This inequality is often called the "union bound". Suppose that a fair coin is tossed n times.

    (i) Using the union bound, find an upper bound on the probability that there is a sequence of $\log_2 n + k$ consecutive heads.

    (ii) What is the expected number of sequences of r consecutive heads?

3 Puzzles/ Misc Problems

13. There are n seats in a plane, labelled 1 through n and there are n people who have been alloted the seats. The first person P_1 sits in a random seat, and subsequently, each person sits in their allotted seat, if it is free, otherwise they pick an empty seat uniformly at random. What is the probability that P_n sits in n?

14. Microsoft's 3-hat puzzle: A team of three people decide on a strategy for playing the following game. Each player walks into a room. On the way in, a fair coin is tossed for each player, deciding that player's hat color, either red or blue. Each player can see the hat colors of the other two players, but cannot see her own hat color. After inspecting each other's hat colors, each player decides on a response, one of: "I have a red hat", "I had a blue hat", or "I pass". The responses are recorded, but the responses are not shared until every player has recorded her response. The team wins if at least one player responds with a color and every color response correctly describes the hat color of the player making the response. In other words, the team loses if either everyone responds with "I pass" or someone responds with a color that is different from her hat color.

    What strategy should one use to maximize the team's chance of winning?

    For example, one possible strategy is to single out one of the three players. This player will respond "I have a red hat" and the others will respond "I pass". The expected chance of winning with this strategy is 50

15. Let A be a (possibly infinite) set of natural numbers. We toss a fair coin until we get heads. Let k be the number of tosses we made. Let's define a random variable X which is 1 if $k \in A$ and which is 0 if $k \notin A$. (i) What is the probability that $X = 1$?

    (ii) Suppose you are given a real number $p \in [0, 1]$ through access to an algorithm that can tell you, for every i, whether the ith bit of p is 1 or 0. That is, when p is written in binary as $p = 0.b_1 b_2 \ldots$, the algorithm can compute b_i for any given i. How can you use this algorithm, along with a fair coin, to simulate a Bernoulli r.v. with $P(X = 1) = p$?

16. (a) Prove that the maximum variance of a random variable in $[0, 1]$ is $1/4$.

    (b) What is the maximum variance that a random variable in a finite interval $[a, b]$ can have, and which distribution achieves it?

17. Prove that for any positive r.v. X, we have: $E[X] \geq e^{E[\log X]}$. Deduce the A.M.-G.M. inequality.