Asymptotic Notation: The Little-Oh

When comparing two functions, we are often able to say something stronger than $f(n) = O(g(n))$. We may find that $g(n)$ grows so much faster than $f(n)$ that the ratio $g(n)/f(n)$ goes to ∞ as n goes to infinity. This is captured in the following definition.

Definition We say that $f(n) = o(g(n))$ if $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = 0$.

For example, we have $5n^2 + 100n - 6 = o(n^3)$. The following connections with the Big-Oh notation are why they are relevant to us.

Proposition 1 *:*

(i) If $f(n) = o(g(n))$, then $f(n) = O(g(n))$.

(ii) If $f(n) = o(g(n))$, then $g(n)$ is NOT $O(f(n))$.

The second part of proposition 1 implies, for example, that $n^3 = O(n^2)$ is false. The implication of the first part is that taking limits of ratios is often a quick and convenient way to compare functions.

The Little-Oh Notation also satisfies the three properties of the Big-Oh, from the second notes: Linearity, Transitivity, and Multiplication on both sides. In fact, it satisfies transitivity even when one of the relations is the weaker Big-Oh.

Proposition 2 *If $f(n) = O(g(n))$ and $g(n) = o(h(n))$, then $f(n) = o(h(n))$.*

If $f(n) = o(g(n))$ and $g(n) = O(h(n))$, then $f(n) = o(h(n))$.