Asymptotic Notation

Asymptotic Upper Bound:

Definition For two functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that $f(x) = O(g(x))$ if there exist $x_0 \in \mathbb{R}$ and $c > 0$ such that for every $x > x_0$, we have:

$$f(x) \leq cg(x).$$

The informal meaning is that the function f grows not faster than g for all sufficiently large x. The same definition holds good for any subdomain of $\mathbb{R}$; in particular, for the analysis of algorithms, we usually consider functions defined on the set of natural numbers.

An example: We shall show that $4n^3 + 100n^2 + 10 = O(n^3)$.

To prove this directly (using the definition), we should find constants n_0 and $c > 0$ such that $4n^3 + 100n^2 + 10 \leq cn^3$ for $n > n_0$. We can easily check that the constants $c = 114$ and $n_0 = 1$ work. Indeed, for $n > 1$, we have $4n^3 = 4n^3$, $100n^2 < 100n^3$ and $10 < 10n^3$. Adding the three inequalities, we get the desired result.

However, such direct proofs and finding explicit constants (c, n_0) is too cumbersome to do all the time, so we will develop a collection of useful results and tricks to compare the growth of two functions.