Asymptotic Notation: Important Examples

Proposition 1 *:*

(i) If f, g are two polynomials of degrees $d_1 < d_2$ respectively, then $f(n) = o(g(n))$. This follows easily from the limit definition.

(ii) $n^k = o(e^n)$ for every constant k. To see this, just note that $e^n \geq \dfrac{n^{k+1}}{(k+1)!}$ (from the Taylor series); thus $n^{k+1} = O(e^n)$ and $n^k = o(n^{k+1})$. Transitivity finishes the proof.

Corollaries:

1. We have $\log n = o(n)$ and $\log \log n = o(\log n)$. To see the first one, substitute $n = e^k$ so that we have $\lim_{n\to\infty} \dfrac{\log n}{n} = \lim_{k\to\infty} \dfrac{k}{e^k}$ Now the latter limit is zero from part (ii) above: it is equivalent to $n = o(e^n)$. The second result is one more substitution.

2. For any constants $c > 1$ and $k > 0$, we have $n^k = o(c^n)$. For proof, let $c > 1$ and write $c = e^\alpha$ for $\alpha > 0$. Then c^n can be written as $e^{\alpha n}$, so that the limit of $n^k/e^\alpha n$ is $1/\alpha^k$ times the limit of $(\alpha n)^k/e^{(\alpha n)}$. This last limit is the same as the one in (ii) but for a substitution, so it is still zero.