Asymptotic Notation: Compare Logarithms

Proposition 1 *Suppose that* $\lim_{n\to\infty} g(n) = \infty$ *and* $\log_2 f(n) = o(\log_2 g(n))$. *Then* $f(n)^a = o(g(n)^b)$ *for* $a, b > 0$.

Examples:

1. $(\log_2 n)^{100} = o(n^{0.01})$.

   Proof: Consider their logarithms $f(n) = 100 \log_2 \log_2 n$ and $g(n) = 0.01 \log_2 n$. We have $f(n) = o(g(n))$, and combining this with Proposition 1 gives the result. It also follows directly from Proposition 2.

2. $n^{\log_2 n} = o(2^{\sqrt{n}})$.

   Proof: Consider their logarithms $f(n) = (\log_2 n)^2$ and $g(n) = \sqrt{n}$. We have $f(n) = o(g(n))$ and applying Proposition 1 gives the result. To see why $f(n) = o(g(n))$, substitute $n = e^k$ so that it is equivalent to $k^2 = o(e^{k/2})$, which we know to be true from Note 4.