$Z_N$ symmetry in $SU(N)$ gauge theories

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\( Z_N \) symmetry plays an important role in the confinement-deconfinement (CD) transition in pure \( SU(N) \) gauge theories. This symmetry is spontaneously broken at high temperature but in presence of matter fields symmetry is broken explicitly.

The non-perturbative studies in 4D show decrease in explicit breaking with the number of temporal lattice points \( (N_\tau) \) i.e in the continuum limit likely there will be reemergence of \( Z_N \) symmetry.[M.Biswal, S.Digal and P.S. Saumia, Nucl.Phys.B 910, 30-39(2016)]

The exact calculation of the partition function to validate the reemergence of \( Z_N \) symmetry is almost impossible in 4D. We attempt to do an exact calculation of partition function after restricting all the spatial gauge fields to zero and matter fields uniform in spatial directions and the problem effectively reduces to a simple temporal 1D model.

The free energy is calculated for a given background of gauge fields and it is independent of the \( Z_N \) explicit breaking term in the large \( N_\tau \) limit i.e the reemergence of \( Z_N \) symmetry in the continuum limit.
The action for a minimally coupled $SU(N)$ gauge theory of fermions and bosons in $3 + 1$ Euclidean space is given by

$$S = \int_V d^3x \int_0^\beta d\tau \left[ \frac{1}{2} \left\{ \text{Tr} (F^{\mu\nu} F_{\mu\nu}) + |D_\mu \Phi|^2 + m_b^2 \Phi^\dagger \Phi \right\} + \bar{\Psi} (\slashed{D} + m_f) \Psi \right]$$

(1)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu], \quad D_\mu \Phi = (\partial_\mu + ig A_\mu) \Phi,$$

$$\slashed{D} \Psi = (\slashed{\partial} + ig \slashed{A}) \Psi$$

Here $A_\mu$, $\Phi$ and $\Psi$ are the gauge, Higgs and the fermion fields respectively, $g$ is the gauge coupling strength, $m_b (m_f)$ is the mass of $\Phi (\Psi)$ fields and $\beta = 1/T$.

The corresponding partition function takes the form

$$Z = \int [DA][D\Phi][D\Phi^\dagger][D\Psi][D\bar{\Psi}] \text{Exp}[-S].$$

(2)
The allowed $A_\mu$ in the path integral are periodic in $\beta$,
$$A_\mu(\vec{x}, \tau = 0) = A_\mu(\vec{x}, \tau = \beta)$$

Pure gauge action and partition function are invariant under the gauge transformation $V(\vec{x}, \tau) \in SU(N)$, $A_\mu$ transforms as
$$A_\mu \rightarrow VA_\mu V^{-1} + \frac{1}{g}(\partial_\mu V)V^{-1}$$

$V(\vec{x}, \tau)$ need not be periodic, as long as it satisfies the following eqn
$$V(\vec{x}, \tau = 0) = zV(\vec{x}, \tau = \beta)$$

where $z \in \mathbb{Z}_N$, with $z = \exp(\frac{2\pi in}{N})$, $n = 0, 1, 2, ..., N - 1$.

So in pure gauge theory the $Z_N$ symmetry is always there which is spontaneously broken above the critical temperature.

The Polyakov loop $L(\vec{x}) = \frac{1}{N} \text{Tr} \left[ P \left\{ \text{Exp} \left( -ig \int_0^\beta A_0 d\tau \right) \right\} \right]$ transforms as $L \rightarrow zL$ under the gauge transformation.
The matter fields contributing to the partition function satisfy the following temporal boundary conditions,

$$\Phi(\vec{x}, \tau = 0) = \Phi(\vec{x}, \tau = \beta)$$

$$\psi(\vec{x}, \tau = 0) = -\psi(\vec{x}, \tau = \beta) \quad (3)$$

These matter fields in the fundamental representation transform under a gauge transformation $V \in SU(N)$ as

$$\phi \rightarrow V\phi, \quad \psi \rightarrow V\psi \quad (4)$$

But the gauge transformed matter fields $(\Phi_g, \Psi_g)$ do not satisfy the boundary condition as mentioned and the $Z_N$ symmetry is broken explicitly.

$$\Phi_g(\vec{x}, \tau = 0) = z\Phi_g(\vec{x}, \tau = \beta)$$

$$\psi_g(\vec{x}, \tau = 0) = z\psi_g(\vec{x}, \tau = \beta) \quad (5)$$
Lattice action in presence of Higgs

The $SU(N)+$Higgs action in 3+1 Euclidean lattice is [M. Biswal, M. Deka, S. Digal and P. S. Saumia, Phys. Rev. D 96, no.1, 014503 (2017)]

$$S = \beta_g \sum_p \left[ 1 - \frac{1}{2} Tr(U_p + U_p^\dagger) \right] - b \sum_{n,\mu} (\Phi_n^\dagger U_{n,\mu} \Phi_{n+\hat{\mu}} + h.c.) + a \sum_n \Phi_n^\dagger \Phi_n. \quad (6)$$

$\beta_g$ is gauge coupling constant, $a = \frac{1}{2}$ and the coupling $b = (m_b^2 + 8)^{-1}$. The plaquette $U_P = U_{n,\hat{\mu}} U_{n+\hat{\nu},\hat{\nu}} U_{n+\hat{\mu},\hat{\mu}} U_{n,\hat{\nu}}$ with gauge link $U_{n,\hat{\mu}}$ and the Higgs mass $m_b$ is expressed in lattice units.

For unit spatial links and $\Phi$ uniform in the spatial directions, the action (along the temporal direction only) reduces to,

$$S = a \sum_{i=1}^{N_\tau} \Phi_i^\dagger \Phi_i - b \sum_{i=1}^{N_\tau} (\Phi_i^\dagger U_i \Phi_{i+1} + h.c.) \quad (7)$$

$\Phi$ satisfies periodic boundary condition, i.e $\Phi_{N_\tau+1} = \Phi_1$. We consider a gauge choice in which $U_i = I$ for $i = 1, 2, ..., N_\tau - 1$ and $U_{N_\tau} = U$. The Polyakov loop is $L = Tr(U)/N$. 

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In order to derive the free energy \( V(L) \), only the \( \Phi_i \) fields in the partition function \( Z_L \) are to be integrated out,

\[
Z_L = \int \prod_{i=1}^{N_\tau} d\Phi_i^\dagger d\Phi_i \ \text{Exp}[-S], \quad S = S_1 + S_2
\]

\[
S_1 = a\Phi_1^\dagger \Phi_1 - b \left( \Phi_{N_\tau}^\dagger U \Phi_1 + \text{h.c.} \right), \quad S_2 = a \sum_{i=2}^{N_\tau} \Phi_i^\dagger \Phi_i - b \sum_{i=1}^{N_\tau-1} (\Phi_i^\dagger \Phi_{i+1} + \text{h.c.})
\]

At first, the fields \( \Phi_2 \) to \( \Phi_{N_\tau-1} \) are integrated out sequentially using Gaussian integration, i.e

\[
Z = \int \prod_{i=2}^{N_\tau-1} d\Phi_i^\dagger d\Phi_i \ \text{Exp}[-S_2]
\]

\[
Z_L = \int d\Phi_1^\dagger d\Phi_{N_\tau}^\dagger d\Phi_{N_\tau} (Z \times \text{Exp}[-S_1])
\]
The partition function after integrating out $\Phi_2$ to $\Phi_{N-1}$ is,

$$Z_L = Q \int d\Phi_1^\dagger d\Phi_1 d\Phi_{N-1}^\dagger d\Phi_{N-1} \times \text{Exp} \left[-A_{N-1} \Phi_{N-1}^\dagger \Phi_{N-1} - C_{N-1} \Phi_1^\dagger \Phi_1 + \left(\Phi_{N-1}^\dagger (B_{N-1} I + bU) \Phi_1 + \text{H.C.}\right)\right]$$

$$Q = \prod_{k=2}^{N-1} l_k^n, \ n = 2N, \ n \text{ is number of components of the } \Phi \text{ field and } C_{N-1} = a - E_{N-1}. \text{ The coefficients } A_{N-1} \text{ to } E_{N-1} \text{ can be obtained by recursion}$$

$$l_{k+1} = \sqrt{\frac{\pi}{A_k}}, \ A_{k+1} = a - \frac{b^2}{A_k}, \ B_{k+1} = \frac{b B_k}{A_k}, \ E_{k+1} = E_k + \frac{B_k^2}{A_k}$$

with $l_2 = 1, A_2 = a, B_2 = b$ and $E_2 = 0$

- After the integration of the remaining fields $\Phi_1$ and $\Phi_{N-1}$ the partition function takes the form,

$$Z_L = Q \sqrt{\frac{\pi^8}{\text{Det}(M)}}$$
Matrix $M$ for $N = 2$

Here the matrix $M$ is $(4N \times 4N)$ given by,

$$
\begin{pmatrix}
A_{N\tau} & B_{N\tau} + bU \\
B_{N\tau} + bU^\dagger & C_{N\tau}
\end{pmatrix}
$$

we consider $N = 2$ and evaluate $Z_L$ explicitly for an arbitrary $U$,

$$U = \alpha_0 + i\alpha.\sigma, \ \alpha = (\alpha_1, \alpha_2, \alpha_3)$$

(8)

where $\sigma_i$’s are the Pauli matrices. The corresponding matrix $M$ is,

$$
M = 
\begin{pmatrix}
A_{N\tau} & 0 & 0 & 0 & B_1 & b\alpha_3 & -b\alpha_2 & b\alpha_1 \\
0 & A_{N\tau} & 0 & 0 & -b\alpha_3 & B_1 & -b\alpha_1 & -b\alpha_2 \\
0 & 0 & A_{N\tau} & 0 & b\alpha_2 & b\alpha_1 & B_1 & -b\alpha_3 \\
0 & 0 & 0 & A_{N\tau} & -b\alpha_1 & b\alpha_2 & b\alpha_3 & B_1 \\
B_1 & -b\alpha_3 & b\alpha_2 & -b\alpha_1 & C_{N\tau} & 0 & 0 & 0 \\
b\alpha_3 & B_1 & b\alpha_1 & b\alpha_2 & 0 & C_{N\tau} & 0 & 0 \\
-b\alpha_2 & -b\alpha_1 & B_1 & b\alpha_3 & 0 & 0 & C_{N\tau} & 0 \\
b\alpha_1 & -b\alpha_2 & -b\alpha_3 & B_1 & 0 & 0 & 0 & C_{N\tau}
\end{pmatrix}
$$

where $B_1 = -(b\alpha_0 + B_{N\tau})$. 

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The determinant of $M$ is,

$$\text{Det}M = \left( B_{N_\tau}^2 - A_{N_\tau} C_{N_\tau} + 2bB_{N_\tau} \alpha_0 + b^2 \right)^4$$

(9)

$Z_2$ rotation of the Polyakov loop ($L = \alpha_0$) changes $\alpha_0 \rightarrow -\alpha_0$. So in the determinant the explicit symmetry breaking of $Z_2$ is $2bB_{N_\tau} \alpha_0$.

For a fixed temperature and physical Higgs mass it is observed that $B_{N_\tau}$ rapidly decreases, vanishing in the larger $N_\tau$ limit restoring the $Z_2$ symmetry. This happens even when the lattice Higgs mass scales as $m_b \propto 1/N_\tau$.

Even for higher $N$ one can see the realisation of $Z_N$ symmetry as the off diagonal elements of the matrix $M$ turn out to be just $U$ and $U^\dagger$ due to vanishing of $B_{N_\tau}$. Effecting a $Z_N$ transformation, i.e $U \rightarrow zU$, the factor $z$ in $U$ will cancel with $z^*$ in $U^\dagger$ leaving the determinant unchanged.
The lattice action for $SU(N)$ staggered fermions is given by

$$S = \beta_g \sum_p \left[ 1 - \frac{1}{2} Tr(U_p + U_p^\dagger) \right] + 2m_f \sum_n \bar{\Psi}_n \Psi_n$$

$$+ \sum_{n,\mu} \eta_{n,\mu} \left[ \bar{\Psi}_n U_{n,\mu} \Psi_{n+\mu} - \bar{\Psi}_n U^\dagger_{n-\mu,\mu} \Psi_{n-\mu} \right] \quad (10)$$

Here the fermion mass as well as the fields are expressed in lattice unit. For unit spatial links and $\Psi$ uniform in the spatial directions, the action (along the temporal direction only) reduces to,

$$S = 2m_f \sum_{i=1}^{N_\tau} \bar{\Psi}_i \Psi_i + \sum_{i=1}^{N_\tau-1} \left( \bar{\Psi}_i \Psi_{i+1} - \bar{\Psi}_{i+1} \Psi_i \right) - \bar{\Psi}_{N_\tau} U \Psi_1 + \bar{\Psi}_1 U^\dagger \Psi_{N_\tau} \quad (11)$$

Considering $\Psi$ satisfies anti-periodic boundary condition, i.e $\Psi_{N_\tau+1} = -\Psi_1$.

Here we have considered the KS phase $\eta_0$ as $+1$ [L. Susskind, Phys. Rev. D 16, 3031-3039 (1977)], however the results/conclusions do not depend on $\eta_0$. 

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To find out the free energy \( V(L) \) we need to integrate out only the fermion fields using standard Grassman integration,

\[
S_1 = 2m_f \bar{\Psi}_1 \Psi_1 - \bar{\Psi}_{N_\tau} U \Psi_1 + \bar{\Psi}_1 U^\dagger \Psi_{N_\tau},
\]

\[
S_2 = 2m_f \sum_{i=2}^{N_\tau} \bar{\Psi}_i \Psi_i + \sum_{i=1}^{N_\tau-1} (\bar{\Psi}_i \Psi_{i+1} - \bar{\Psi}_{i+1} \Psi_i).
\] (12)

Initially we integrate the fields \( \Psi_2, \bar{\Psi}_2 \) to \( \Psi_{N_\tau-1}, \bar{\Psi}_{N_\tau-1} \) sequentially just as in the previous section using Grassman integration. Afterwards \( \Psi_1, \bar{\Psi}_1 \) and \( \Psi_{N_\tau}, \bar{\Psi}_{N_\tau} \) are integrated out to obtain the partition function,

\[
Z_L = \int d\bar{\Psi}_1 d\bar{\Psi}_{N_\tau} d\Psi_1 d\Psi_{N_\tau} \exp[-S_1] Z, \tag{13}
\]

\[
Z = \int \prod_{i=2}^{N_\tau-1} d\bar{\Psi}_i d\Psi_i \exp[-S_2].
\]
The partition function can be written as,

$$Z_L = \int d\bar{\Psi}_1 d\psi_1 d\bar{\Psi}_{N_\tau} d\psi_{N_\tau} \exp \left[ \bar{\Psi}_{N_\tau} U \Psi_1 - \bar{\Psi}_1 U^\dagger \Psi_{N_\tau} \right] \times$$

$$\prod_r \left( 1 - 2m_f \bar{\psi}_1^r \psi_1^r - 2m_f \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r + 4m_f^2 \bar{\psi}_1^r \psi_1^r \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r \right) \times$$

$$(A_{N_\tau} - B_{N_\tau} \bar{\psi}_1^r \psi_1^r - C_{N_\tau} \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r + \bar{\psi}_{N_\tau}^r \psi_1^r + D_{N_\tau} \bar{\psi}_1^r \psi_{N_\tau}^r + E_{N_\tau} \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r \bar{\psi}_1^r \psi_1^r).$$

Note that $\psi_i^r$ denotes the colour $r$ of the field $\Psi_i$ at the temporal site $i$. The coefficients $A_{N_\tau}$ to $E_{N_\tau}$ can be obtained by recursion as,

$$A_{k+1} = 2m_f A_k + C_k, \quad B_{k+1} = 2m_f B_k + E_k, \quad C_{k+1} = A_k,$$

$$D_{k+1} = (-1)^k, \quad E_{k+1} = B_k \quad (14)$$

with $A_4 = (1 + 4m_f^2), \quad B_4 = 2m_f, \quad C_4 = 2m_f, \quad E_4 = 1,$
Partition function for $N = 2$ and $Z_2$ symmetry

The simplified partition function is,

$$\mathcal{Z}_L = \int d\bar{\Psi}_1 d\Psi_1 d\bar{\Psi}_{N_t} d\Psi_{N_t} \ \text{Exp} \left[ \bar{\Psi}_{N_t} U \Psi_1 - \bar{\Psi}_1 U^\dagger \Psi_{N_t} \right] \times \prod_r \left( \tilde{A} - \tilde{B} \bar{\psi}_1^{r'} \psi_1^r - \tilde{C} \bar{\psi}_{N_t}^{r'} \psi_{N_t}^{r'} + \bar{\psi}_{N_t}^{r'} \psi_1^{r'} + \tilde{D} \bar{\psi}_1^{r'} \psi_{N_t}^{r'} + \tilde{E} \bar{\psi}_{N_t}^{r'} \psi_{N_t}^{r'} \bar{\psi}_1^{r'} \psi_1^{r'} \right)$$

(15)

where $\tilde{A} = A_{N_t}$, $\tilde{B} = (2m_f A_{N_t} + B_{N_t})$, $\tilde{C} = (2m_f A_{N_t} + C_{N_t})$, $\tilde{D} = D_{N_t}$ and $\tilde{E} = E_{N_t} + 2m_f C_{N_t} + 2m_f B_{N_t} + 4m_f^2 A_{N_t}$.

For $N = 2$, integration of the rest of the fields in Eq.15 leads to,

$$\mathcal{Z}_L = \tilde{E}^2 + 2\tilde{E} \tilde{A} |U_{11}|^2 + \tilde{A}^2 + 2\tilde{B} \tilde{C} |U_{12}|^2 + 2(1 - \tilde{D} \text{Re}(U_{11}^2)) + (\tilde{E} + \tilde{A})(1 - \tilde{D}) \text{tr}(U).$$

(16)

For non zero $m_f$, in the free energy $V(L)$ the first four terms of $\mathcal{Z}_L$ dominate over $\tilde{E} + \tilde{A}$. The dominance only grows with $N_{\tau}$, hence in the limit of large $N_{\tau}$ the $Z_2$ symmetry is realized.
Partition function for $N > 2$

- For higher $N$ it is difficult to evaluate $Z_L$ for a general $U$. To proceed further we assume the $U$ to be $U_{rs} = \lambda_r \delta_{rs}$. After the exponential term in partition function is written as a polynomial,

$$Z_L = \int d\bar{\Psi}_1 d\Psi_1 d\bar{\Psi}_{N\tau} d\Psi_{N\tau} \times$$

$$\prod_r \left( A - B\bar{\psi}_1^r \psi_1^r - C\bar{\psi}_{N\tau}^r \psi_{N\tau}^r + F_r \bar{\psi}_{N\tau}^r \psi_1^r + D_r \bar{\psi}_1^r \psi_{N\tau}^r + E_r \bar{\psi}_{N\tau}^r \psi_{N\tau}^r \bar{\psi}_1^r \psi_1^r \right)$$

where $A = \tilde{A}$, $B = \tilde{B}$, $C = \tilde{C}$, $D_r = \tilde{D} - \lambda_r^* \tilde{A}$, $E_r = \tilde{E} - \lambda_r \tilde{D} + \lambda_r^* + \tilde{A}$ and $F_r = (1 + \lambda_r \tilde{A})$

- The partition function for higher $N$ is,

$$Z_L = \prod_r E_r \quad \text{(17)}$$
The free energy is,

\[ V(L) = -T \sum_r \left\{ \log (\tilde{E} + \tilde{A}) + \log \left( 1 - \frac{\lambda_r \tilde{D} - \lambda_r^*}{\tilde{E} + \tilde{A}} \right) \right\} \quad (18) \]

To see the realization of \( Z_N \) symmetry for a fixed temperature and physical fermion mass, the behaviour of \( \tilde{E} \) and \( \tilde{A} \) is studied in the limit \( N_\tau \to \infty \) while scaling the fermion mass in lattice units as \( m_f \propto 1/N_\tau \).

We have numerically checked that \( \tilde{E} \) and \( \tilde{A} \) monotonically increase with \( N_\tau \) even when the lattice fermion mass scales as \( m_f \propto 1/N_\tau \). The increase though is slower compared to the case when \( m_f \) is held fixed. The \( Z_N \) explicit breaking decreases with increase in \( N_\tau \).
The vanishingly small explicit breaking of $Z_N$ for $N_\tau \to \infty$ can be attributed to the dominance of the density of the states over the action. In this limit, the density of states is found to have the $Z_N$ symmetry. [M. Biswal, S. Digal, V. Mamale and S. Shaikh, Mod. Phys. Lett. A 36, no.30, 2150218 (2021)]

The spatial links as well as the spatial modes of the matter fields determine the boundaries separating regions where $Z_N$ symmetry is realised from the rest. These modes are responsible for the Higgs and the chiral transitions, which are entropy driven.

It is expected that in the phase diagram where the action dominates over the entropy the $Z_N$ symmetry will be explicitly broken.

Recent Monte Carlo simulations in the presence of Higgs show that this is indeed the case. The $Z_N$ symmetry is explicitly broken in the Higgs broken phase even for large $N_\tau$. 
Thank you for your attention!