Z_N symmetry in $SU(N)$ gauge theories

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Outline

Introduction

- Z_N symmetry plays an important role in the confinement-deconfinement (CD) transition in pure $SU(N)$ gauge theories. This symmetry is spontaneously broken at high temperature but in presence of matter fields symmetry is broken explicitly.
- The non-perturbative studies in 4D show decrease in explicit breaking with the number of temporal lattice points (N_τ) i.e in the continuum limit likely there will be reemergence of Z_N symmetry.[M.Biswal, S.Digal and P.S. Saumia, Nucl.Phys.B 910, 30-39(2016)]
- The exact calculation of the partition function to validate the reemergence of Z_N symmetry is almost impossible in 4D. We attempt to do an exact calculation of partition function after restricting all the spatial gauge fields to zero and matter fields uniform in spatial directions and the problem effectively reduces to a simple temporal 1D model.
- The free energy is calculated for a given background of gauge fields and it is independent of the Z_N explicit breaking term in the large N_τ limit i.e the reemergence of Z_N symmetry in the continuum limit.

Total action and partition function

- The action for a minimally coupled $SU(N)$ gauge theory of fermions and bosons in $3+1$ Euclidean space is given by

$$S = \int_V d^3x \int_0^\beta d\tau \left[\frac{1}{2} \left\{ Tr\left(F^{\mu\nu} F_{\mu\nu}\right) + |D_\mu \Phi|^2 + m_b^2 \Phi^\dagger \Phi \right\} + \bar{\Psi}(\not{D} + m_f)\Psi \right] \quad (1)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu],\ D_\mu \Phi = (\partial_\mu + igA_\mu)\Phi,$$

$$\not{D}\Psi = (\not{\partial} + ig\not{A})\Psi$$

Here A_μ, Φ and Ψ are the gauge, Higgs and the fermion fields respectively, g is the gauge coupling strength, $m_b(m_f)$ is the mass of $\Phi(\Psi)$ fields and $\beta = 1/T$.

- The corresponding partition function takes the form

$$\mathcal{Z} = \int [DA][D\Phi][D\Phi^\dagger][D\Psi][D\bar{\Psi}]\ \mathrm{Exp}[-\mathrm{S}]. \quad (2)$$

Z_N symmetry in pure $SU(N)$ gauge theory

- The allowed A_μ in the path integral are periodic in β,
$$A_\mu(\vec{x}, \tau = 0) = A_\mu(\vec{x}, \tau = \beta)$$

- Pure gauge action and partition function are invariant under the gauge transformation $V(\vec{x}, \tau) \in SU(N)$, A_μ transforms as
$$A_\mu \longrightarrow V A_\mu V^{-1} + \frac{1}{g}(\partial_\mu V) V^{-1}$$
$V(\vec{x}, \tau)$ need not be periodic, as long as it satisfies the following eqn
$$V(\vec{x}, \tau = 0) = zV(\vec{x}, \tau = \beta)$$
where $z \in Z_N$, with $z = \mathbb{I} exp(\frac{2\pi i n}{N})$, $n = 0, 1, 2, ..., N-1$.

- So in pure gauge theory the Z_N symmetry is always there which is spontaneously broken above the critical temperature.

- The Polyakov loop $L(\vec{x}) = \frac{1}{N}\mathrm{Tr}\left[\mathrm{P}\left\{\mathrm{Exp}\left(-ig\int_0^\beta A_0 d\tau\right)\right\}\right]$ transforms as $L \longrightarrow zL$ under the gauge transformation.

Explicit symmetry breaking in presence of matter fields

- The matter fields contributing to the partition function satisfy the following temporal boundary conditions,

$$\Phi(\vec{x}, \tau = 0) = \Phi(\vec{x}, \tau = \beta)$$

$$\Psi(\vec{x}, \tau = 0) = -\Psi(\vec{x}, \tau = \beta) \tag{3}$$

- These matter fields in the fundamental representation transform under a gauge transformation $V \in SU(N)$ as

$$\Phi \longrightarrow V\Phi, \ \ \Psi \longrightarrow V\Psi \tag{4}$$

- But the gauge transformed matter fields (Φ_g, Ψ_g) do not satisfy the boundary condition as mentioned and the Z_N symmetry is broken explicitly.

$$\Phi_g(\vec{x}, \tau = 0) = z\Phi_g(\vec{x}, \tau = \beta)$$

$$\Psi_g(\vec{x}, \tau = 0) = z\Psi_g(\vec{x}, \tau = \beta) \tag{5}$$

Lattice action in presence of Higgs

- The $SU(N)$+Higgs action in 3+1 Euclidean lattice is [M. Biswal, M. Deka, S. Digal and P. S. Saumia, Phys. Rev. D 96, no.1, 014503 (2017)]

$$S = \beta_g \sum_p \left[1 - \frac{1}{2} Tr(U_p + U_p^\dagger)\right] - b \sum_{n,\mu} (\Phi_n^\dagger U_{n,\mu} \Phi_{n+\hat{\mu}} + h.c) + a \sum_n \Phi_n^\dagger \Phi_n. \quad (6)$$

β_g is gauge coupling constant, $a = \frac{1}{2}$ and the coupling $b = (m_b^2 + 8)^{-1}$.
The plaquette $U_P = U_{n,\hat{\mu}} U_{n+\hat{\mu},\hat{\nu}} U^\dagger_{n+\hat{\nu},\hat{\mu}} U^\dagger_{n,\hat{\nu}}$ with gauge link $U_{n,\hat{\mu}}$ and the Higgs mass m_b is expressed in lattice units.

- For unit spatial links and Φ uniform in the spatial directions, the action (along the temporal direction only) reduces to,

$$S = a \sum_{i=1}^{N_\tau} \Phi_i^\dagger \Phi_i - b \sum_{i=1}^{N_\tau} (\Phi_i^\dagger U_i \Phi_{i+1} + h.c.) \quad (7)$$

Φ satisfies periodic boundary condition, i.e $\Phi_{N_\tau+1} = \Phi_1$.
We consider a gauge choice in which $U_i = \mathbb{I}$ for $i = 1, 2, ..., N_\tau - 1$ and $U_{N_\tau} = U$. The Polyakov loop is $L = Tr(U)/N$.

Partition function in reduced temporal 1D chain

- In order to derive the free energy $V(L)$, only the Φ_i fields in the partition function $\mathcal{Z}_L$ are to be integrated out,

$$\mathcal{Z}_L = \int \prod_{i=1}^{N_\tau} d\Phi_i^\dagger d\Phi_i \ \mathrm{Exp}[-S], \ S = S_1 + S_2$$

$$S_1 = a\Phi_1^\dagger \Phi_1 - b\left(\Phi_{N_\tau}^\dagger U \Phi_1 + h.c.\right), \ S_2 = a\sum_{i=2}^{N_\tau} \Phi_i^\dagger \Phi_i - b\sum_{i=1}^{N_\tau - 1} (\Phi_i^\dagger \Phi_{i+1} + h.c.)$$

- At first, the fields Φ_2 to $\Phi_{N_\tau - 1}$ are integrated out sequentially using Gaussian integration, i.e

$$\mathcal{Z} = \int \prod_{i=2}^{N_\tau - 1} d\Phi_i^\dagger d\Phi_i \ \mathrm{Exp}[-S_2] \tag{8}$$

$$\mathcal{Z}_L = \int d\Phi_1^\dagger d\Phi_1 d\Phi_{N_\tau}^\dagger d\Phi_{N_\tau} \left(\mathcal{Z} \times \mathrm{Exp}[-S_1]\right)$$

Exact partition function

The partition function after integrating out Φ_2 to $\Phi_{N_\tau - 1}$ is,

$$\mathcal{Z}_L = Q \int d\Phi_1^\dagger d\Phi_1 d\Phi_{N_\tau}^\dagger d\Phi_{N_\tau} \times$$
$$\mathrm{Exp}\left[-\mathrm{A}_{\mathrm{N}_\tau}\Phi^\dagger_{\mathrm{N}_\tau}\Phi_{\mathrm{N}_\tau} - \mathrm{C}_{\mathrm{N}_\tau}\Phi_1^\dagger\Phi_1 + \left(\Phi^\dagger_{\mathrm{N}_\tau}(\mathrm{B}_{\mathrm{N}_\tau}\mathbb{I} + \mathrm{bU})\Phi_1 + \mathrm{H.C.}\right)\right]$$

$Q = \prod_{k=2}^{N_\tau} I_k^n$, $n = 2N$, n is number of components of the Φ field and $C_{N_\tau} = a - E_{N_\tau}$. The coefficients A_{N_τ} to E_{N_τ} can be obtained by recursion

$$I_{k+1} = \sqrt{\frac{\pi}{A_k}},\ A_{k+1} = a - \frac{b^2}{A_k},\ B_{k+1} = \frac{bB_k}{A_k},\ E_{k+1} = E_k + \frac{B_k^2}{A_k}$$

with $I_2 = 1,\ A_2 = a,\ B_2 = b$ and $E_2 = 0$

- After the integration of the remaining fields Φ_1 and Φ_{N_τ} the partition function takes the form,

$$\mathcal{Z}_L = Q\sqrt{\frac{\pi^8}{Det(M)}}$$

Matrix M for $N = 2$

Here the matrix M is $(4N \times 4N)$ given by,

$$\begin{pmatrix} A_{N_\tau} & B_{N_\tau} + bU \\ B_{N_\tau} + bU^\dagger & C_{N_\tau} \end{pmatrix}$$

we consider $N = 2$ and evaluate $\mathcal{Z}_L$ explicitly for an arbitrary U,

$$U = \alpha_0 + i\alpha.\sigma, \ \ \alpha = (\alpha_1, \alpha_2, \alpha_3) \quad (8)$$

where σ_i's are the Pauli matrices. The corresponding matrix M is,

$$\text{M}=\begin{pmatrix} A_{N_\tau} & 0 & 0 & 0 & B_1 & b\alpha_3 & -b\alpha_2 & b\alpha_1 \\ 0 & A_{N_\tau} & 0 & 0 & -b\alpha_3 & B_1 & -b\alpha_1 & -b\alpha_2 \\ 0 & 0 & A_{N_\tau} & 0 & b\alpha_2 & b\alpha_1 & B_1 & -b\alpha_3 \\ 0 & 0 & 0 & A_{N_\tau} & -b\alpha_1 & b\alpha_2 & b\alpha_3 & B_1 \\ B_1 & -b\alpha_3 & b\alpha_2 & -b\alpha_1 & C_{N_\tau} & 0 & 0 & 0 \\ b\alpha_3 & B_1 & b\alpha_1 & b\alpha_2 & 0 & C_{N_\tau} & 0 & 0 \\ -b\alpha_2 & -b\alpha_1 & B_1 & b\alpha_3 & 0 & 0 & C_{N_\tau} & 0 \\ b\alpha_1 & -b\alpha_2 & -b\alpha_3 & B_1 & 0 & 0 & 0 & C_{N_\tau} \end{pmatrix},$$

where $B_1 = -(b\alpha_0 + B_{N_\tau})$.

Realization of Z_N symmetry

- The determinant of M is,

$$DetM = \left(B_{N_\tau}^2 - A_{N_\tau} C_{N_\tau} + 2bB_{N_\tau}\alpha_0 + b^2\right)^4 \tag{9}$$

- Z_2 rotation of the Polyakov loop($L = \alpha_0$) changes $\alpha_0 \rightarrow -\alpha_0$. So in the determinant the explicit symmetry breaking of Z_2 is $2bB_{N_\tau}\alpha_0$.
- For a fixed temperature and physical Higgs mass it is observed that B_{N_τ} rapidly decreases, vanishing in the larger N_τ limit restoring the Z_2 symmetry.This happens even when the lattice Higgs mass scales as $m_b \propto 1/N_\tau$.
- Even for higher N one can see the realisation of Z_N symmetry as the off diagonal elements of the matrix M turn out to be just U and $U^\dagger$ due to vanishing of B_{N_τ}. Effecting a Z_N transformation, i.e $U \rightarrow zU$, the factor z in U will cancel with z^* in $U^\dagger$ leaving the determinant unchanged.

Lattice action in presence of fermion

- The lattice action for $SU(N)$ staggered fermions is given by [G. W. Kilcup and S. R. Sharpe, Nucl. Phys. B 283, 493-550 (1987)]

$$S = \beta_g \sum_p \left[1 - \frac{1}{2} Tr(U_p + U_p^\dagger)\right] + 2m_f \sum_n \bar{\Psi}_n \Psi_n + \sum_{n,\mu} \eta_{n,\mu} \left[\bar{\Psi}_n U_{n,\mu} \Psi_{n+\mu} - \bar{\Psi}_n U^\dagger_{n-\mu,\mu} \Psi_{n-\mu}\right] \quad (10)$$

Here the fermion mass as well as the fields are expressed in lattice unit.
For unit spatial links and Ψ uniform in the spatial directions, the action (along the temporal direction only) reduces to,

$$S = 2m_f \sum_{i=1}^{N_\tau} \bar{\Psi}_i \Psi_i + \sum_{i=1}^{N_\tau - 1} \left(\bar{\Psi}_i \Psi_{i+1} - \bar{\Psi}_{i+1} \Psi_i\right) - \bar{\Psi}_{N_\tau} U \Psi_1 + \bar{\Psi}_1 U^\dagger \Psi_{N_\tau} \quad (11)$$

Considering Ψ satisfies anti-periodic boundary condition, i.e $\Psi_{N_\tau+1} = -\Psi_1$.

- Here we have considered the KS phase η_0 as $+1$ [L. Susskind, Phys. Rev. D 16, 3031-3039 (1977)], however the results/conclusions do not depend on η_0.

Partition function in reduced temporal 1D chain

- To find out the free energy $V(L)$ we need to integrate out only the fermion fields using standard Grassman integration,

$$S_1 = 2m_f \bar{\Psi}_1 \Psi_1 - \bar{\Psi}_{N_\tau} U \Psi_1 + \bar{\Psi}_1 U^\dagger \Psi_{N_\tau}, \qquad (12)$$

$$S_2 = 2m_f \sum_{i=2}^{N_\tau} \bar{\Psi}_i \Psi_i + \sum_{i=1}^{N_\tau - 1} \left(\bar{\Psi}_i \Psi_{i+1} - \bar{\Psi}_{i+1} \Psi_i \right).$$

- Initially we integrate the fields Ψ_2, $\bar{\Psi}_2$ to $\Psi_{N_\tau - 1}$, $\bar{\Psi}_{N_\tau - 1}$ sequentially just as in the previous section using Grassman integration. Afterwards Ψ_1, $\bar{\Psi}_1$ and Ψ_{N_τ}, $\bar{\Psi}_{N_\tau}$ are integrated out to obtain the partition function,

$$\mathcal{Z}_L = \int d\bar{\Psi}_1 d\bar{\Psi}_{N_\tau} d\Psi_1 d\Psi_{N_\tau} \, \mathrm{Exp}[-S_1] \mathcal{Z}, \qquad (13)$$

$$\mathcal{Z} = \int \prod_{i=2}^{N_\tau - 1} d\bar{\Psi}_i d\Psi_i \, \mathrm{Exp}[-S_2].$$

Co-efficients

The partition function can be written as,

$$\mathcal{Z}_L = \int d\bar{\Psi}_1 d\Psi_1 d\bar{\Psi}_{N_\tau} d\Psi_{N_\tau} \ \mathrm{Exp}\left[\bar{\Psi}_{\mathrm{N}_\tau} \mathrm{U} \Psi_1 - \bar{\Psi}_1 \mathrm{U}^\dagger \Psi_{\mathrm{N}_\tau}\right] \times$$
$$\prod_r \left(1 - 2m_f \bar{\psi}_1^r \psi_1^r - 2m_f \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r + 4m_f^2 \bar{\psi}_1^r \psi_1 \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r\right) \times$$
$$\left(A_{N_\tau} - B_{N_\tau} \bar{\psi}_1^r \psi_1^r - C_{N_\tau} \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r + \bar{\psi}_{N_\tau}^r \psi_1^r + D_{N_\tau} \bar{\psi}_1^r \psi_{N_\tau}^r + E_{N_\tau} \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r \bar{\psi}_1^r \psi_1^r\right).$$

Note that ψ_i^r denotes the colour r of the field Ψ_i at the temporal site i. The coefficients A_{N_τ} to E_{N_τ} can be obtained by recursion as,

$$A_{k+1} = 2m_f A_k + C_k,\ B_{k+1} = 2m_f B_k + E_k,\ C_{k+1} = A_k,$$
$$D_{k+1} = (-1)^k,\ E_{k+1} = B_k \quad (14)$$

with $A_4 = (1 + 4m_f^2),\ B_4 = 2m_f,\ C_4 = 2m_f,\ E_4 = 1,$

Partition function for $N = 2$ and Z_2 symmetry

- The simplified partition function is,

$$\mathcal{Z}_L = \int d\bar{\Psi}_1 d\Psi_1 d\bar{\Psi}_{N_\tau} d\Psi_{N_\tau} \ \mathrm{Exp}\left[\bar{\Psi}_{\mathrm{N}_\tau} \mathrm{U} \Psi_1 - \bar{\Psi}_1 \mathrm{U}^\dagger \Psi_{\mathrm{N}_\tau}\right] \times$$
$$\prod_r \left(\tilde{A} - \tilde{B}\bar{\psi}_1^r \psi_1^r - \tilde{C}\bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r + \bar{\psi}_{N_\tau}^r \psi_1^r + \tilde{D}\bar{\psi}_1^r \psi_{N_\tau}^r + \tilde{E}\bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r \bar{\psi}_1^r \psi_1^r\right) \quad (15)$$

where $\tilde{A} = A_{N_\tau}$, $\tilde{B} = (2m_f A_{N_\tau} + B_{N_\tau})$, $\tilde{C} = (2m_f A_{N_\tau} + C_{N_\tau})$, $\tilde{D} = D_{N_\tau}$ and $\tilde{E} = E_{N_\tau} + 2m_f C_{N_\tau} + 2m_f B_{N_\tau} + 4m_f^2 A_{N_\tau}$.

- For $N = 2$, integration of the rest of the fields in Eq.15 leads to,

$$\begin{aligned} \mathcal{Z}_L \quad &= \tilde{E}^2 + 2\tilde{E}\tilde{A}|U_{11}|^2 + \tilde{A}^2 + 2\tilde{B}\tilde{C}|U_{12}|^2 + 2(1 - \tilde{D}Re(U_{11}^2)) \\ &+(\tilde{E} + \tilde{A})(1 - \tilde{D})tr(U). \end{aligned} \quad (16)$$

For non zero m_f, in the free energy $V(L)$ the first four terms of $\mathcal{Z}_L$ dominate over $\tilde{E} + \tilde{A}$. The dominance only grows with N_τ, hence in the limit of large N_τ the Z_2 symmetry is realized.

Partition function for $N > 2$

- For higher N it is difficult to evaluate $\mathcal{Z}_L$ for a general U. To proceed further we assume the U to be $U_{rs} = \lambda_r \delta_{rs}$. After the exponential term in partition function is written as a polynomial,

$$\mathcal{Z}_L = \int d\bar{\Psi}_1 d\Psi_1 d\bar{\Psi}_{N_\tau} d\Psi_{N_\tau} \times$$
$$\prod_r (A - B\bar{\psi}_1^r \psi_1^r - C\bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r + F_r \bar{\psi}_{N_\tau}^r \psi_1^r + D_r \bar{\psi}_1^r \psi_{N_\tau}^r + E_r \bar{\psi}_{N_\tau}^r \psi_{N_\tau}^r \bar{\psi}_1^r \psi_1^r)$$

where $A = \tilde{A}$, $B = \tilde{B}$, $C = \tilde{C}$, $D_r = \tilde{D} - \lambda_r^* \tilde{A}$, $E_r = \tilde{E} - \lambda_r \tilde{D} + \lambda_r^* + \tilde{A}$ and $F_r = (1 + \lambda_r \tilde{A})$

- The partition function for higher N is,

$$\mathcal{Z}_L = \prod_r E_r \tag{17}$$

Free energy for $N > 2$ and Z_N symmetry

- The free energy is,

$$V(L) = -T\sum_r \left\{ \log\left(\tilde{\mathrm{E}} + \tilde{\mathrm{A}}\right) + \log\left(1 - \frac{\lambda_\mathrm{r}\tilde{\mathrm{D}} - \lambda_\mathrm{r}^*}{\tilde{\mathrm{E}} + \tilde{\mathrm{A}}}\right)\right\} \qquad (18)$$

- To see the realization of Z_N symmetry for a fixed temperature and physical fermion mass, the behaviour of $\tilde{E}$ and $\tilde{A}$ is studied in the limit $N_\tau \to \infty$ while scaling the fermion mass in lattice units as $m_f \propto 1/N_\tau$.

- We have numerically checked that $\tilde{E}$ and $\tilde{A}$ monotonically increase with N_τ even when the lattice fermion mass scales as $m_f \propto 1/N_\tau$. The increase though is slower compared to the case when m_f is held fixed. The Z_N explicit breaking decreases with increase in N_τ.

Summary

- The vanishingly small explicit breaking of Z_N for $N_\tau \to \infty$ can be attributed to the dominance of the density of the states over the action. In this limit, the density of states is found to have the Z_N symmetry. [M. Biswal, S. Digal, V. Mamale and S. Shaikh, Mod. Phys. Lett. A 36, no.30, 2150218 (2021)]

- The spatial links as well as the spatial modes of the matter fields determine the boundaries separating regions where Z_N symmetry is realised from the rest.These modes are responsible for the Higgs and the chiral transitions, which are entropy driven.

- It is expected that in the phase diagram where the action dominates over the entropy the Z_N symmetry will be explicitly broken.

- Recent Monte Carlo simulations in the presence of Higgs show that this is indeed the case. The Z_N symmetry is explicitly broken in the Higgs broken phase even for large N_τ.

Thank you for your attention!