BM4040 Mechanobiology

Network Elasticity

0.1 Derivation of microscopic energy

When the network is applied with a two-dimensional tension T, the length of the spring changes from l_0 to

$$l_t = \frac{l_0}{1 - \frac{T}{\sqrt{3}k}} \tag{1}$$

which we had obtained in class by minimizing the enthalpy

$$H = E - TA = \frac{3}{2}k(l - l_0)^2 - \frac{\sqrt{3}}{2}Tl^2. \tag{2}$$

0.1.1 Biaxial stretching

Then, an infinitesimal deformation of the network stretches the springs further from length l_t to $l_t + \delta$ after the deformation. During this infinitesimal biaxial stretching deformation, the change in the enthalpy is

$$\Delta H = \underbrace{\frac{3}{2}k(l_t + \delta - l_0)^2 - \frac{\sqrt{3}}{2}T(l_t + \delta)^2}_{\text{after infinitesimal deformation}} - \underbrace{\frac{3}{2}k(l_t - l_0)^2 - \frac{\sqrt{3}}{2}Tl_t^2}_{\text{before infinitesimal deformation}} \tag{3}$$

We substitute $l_0 = l_t\left(1 - \frac{T}{\sqrt{3}k}\right)$ in the expression for ΔH to obtain

$$\Delta H \approx \frac{3}{2}k\delta^2 - \frac{\sqrt{3}}{2}T\delta^2 \tag{4}$$

This gives the change in the enthalpy per unit area to be

$$\frac{\Delta H}{A} \approx \frac{2}{\sqrt{3}l_t^2}\left(\frac{3}{2}k\delta^2 - \frac{\sqrt{3}}{2}T\delta^2\right) = \sqrt{3}k\left(1 - \frac{T}{\sqrt{3}k}\right)\left(\frac{\delta}{l_t}\right)^2 \tag{5}$$

0.1.2 Simple shear

Following the similar steps as above, we get for simple shear deformation

$$\Delta H = \underbrace{\frac{1}{2}k\left(l_t + \frac{\delta}{2} + \frac{\delta^2}{2l_0} - l_0\right)^2 + \frac{1}{2}k\left(l_t - \frac{\delta}{2} + \frac{\delta^2}{2l_0} - l_0\right)^2 + \frac{1}{2}k\left(l_t - l_0\right)^2}_{\text{after infinitesimal shear}} - \underbrace{\frac{3}{2}k\left(l_t - l_0\right)^2}_{\text{before infinitesimal shear}} \tag{6}$$

This gives after substituting $l_0 = l_t\left(1 - \frac{T}{\sqrt{3}k}\right)$ and simplifying by keeping terms only upto the order of δ^2

$$\frac{\Delta H}{A} \approx \frac{2}{\sqrt{3}l_t^2}\left(\frac{1}{4}k\delta^2 + \frac{\sqrt{3}}{4}T\delta^2\right) = \frac{k}{2\sqrt{3}}\left(1 + \frac{\sqrt{3}T}{k}\right)\left(\frac{\delta}{l_t}\right)^2 \tag{7}$$

0.2 SageMath code

For your convenience, the `SageMath` code snippet for this calculation is given below.
You can evaluate it at https://sagecell.sagemath.org/.

```
# definition of variables
l,lt,l0,d = var('l,l_t,l_0,delta')
k,T = var('k,T')

# enthalpy (biaxial stretch)
H = (3*k/2)*(lt+d-l0)**2 - (sqrt(3)*T/2)*(lt+d)**2
dH = H - H.substitute(d==0)
l0val = lt*(1-T/(sqrt(3)*k))

dH = dH.substitute(l0==l0val).expand()
show(dH)

# enthalpy (shear)
H =  (1*k/2)*(lt+d/2 + d*d/(2*l0)-l0)**2 + (1*k/2)*(lt-d/2 + d*d/(2*l0)-l0)**2 + (1*k/2)*(lt-l0)**2
dH = H - H.substitute(d==0)
dH = dH.substitute(l0==l0val).expand()
show(dH.taylor(d,0,2))
```